IMS

MATHS

BOOK-17

Basic Concepts

Mechanics is a science which deals with the behaviour of material bodies under the action of external forces.

Mechanics is a branch of science which deals with the conditions of rest or motion of bodies relative to their surroundings.

Mechanics is that branch of science which deals with the study of body in state of motion (or) at rest under the effect of some forces.

Mechanics is the scientific bedrock of modern Technology:

The boatman rowing his boat across a stream, the astronomer concerned with the motion of plants, the space technologist concerned with the spaceship, the engineer concerned with the erection of a dam etc, all have to apply mechanics.

Mechanics is a physical science:

Mechanics has its roots in the physical phenomenon which occur in nature and it is sustained by human's desire to predict, to describe, to control and to understand such phenomena.

mathematics it is validity of derivation

→ Mechanics is a formal science.

Mechanics has its own structure. The object of its study are not the real physical objects as they exist but abstract concepts – idealised models of the objects outside.

internal world.

→ Mechanics is an applied science

application of scientific knowledge transferred into physical environment

Mechanics is very much applicable to industry, defence and social sciences. It is an essential tool of the physicist, to the engineer, to the space technologist and to the defence – chief. Hence the assertion that mechanics is an applied science and resembles mathematics.

※→ Development, types and Branches of Mechanics

Development of Mechanics: The development of mechanics as a scientific displine depends upon the following three stages.

(i) Stage of Observation:

We can observe the physical world around us.

for example: we can observe the motion of planets around the sun, motion of the moon around the earth, mutual attraction between physical bodies, earth quakes, tides etc.

(ii) Postulation of laws: we seek the laws which govern the behaviour of objects in our physical world from our observations.

For example: Newton's laws of motion, Newtonian law of gravitation etc.

(iii) Validity and Verification of laws: The results derived out of mathematical theory (developed out of the postulation of laws) are compared experimentally with the observations made. When the results of theory and experiment are identical, the theory can be used to build dams, bridges, to launch Appolo to moon etc. When the results of theory and experiment are not identical, new laws are postulated and new theories are developed.

In 1905 A.D., A Einstein placed limitations on Newton laws of motion in his theory of relativity and thus set the stage for the development of relativistic mechanics and Quantum mechanics.

Types of Mechanics

(i) Newtonian Mechanics: This is based on the application of Newton's laws of motion directly. This is also known as classical mechanics or Analytical Mechanics.

Historically, Mechanics was the earliest branch of physics to be developed as an exact science. It was left to Galileo (1564 – 1642) and Newton

Forces in this system are also called forces in space.
the force system may further be classified as follows

→ **Collinear Force System:**

This system exists if all the forces act along the same line of action.

the system is always Coplanar.

→ **Concurrent force System:** this system exists if all lines of action of the forces intersect at a point

This system may be coplanar or non-coplanar.

Note: Concurrent force system of the two forces is always coplanar.

This is because a plane can be made through two concurrent straight lines.

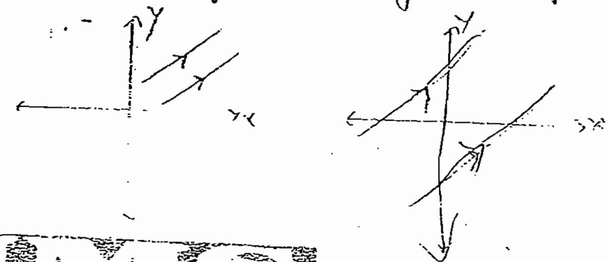
→ **Non-Concurrent force System:**

this system exists if the lines of action of the forces donot meet at one point.

This system may be Coplanar or non-Coplanar.

→ **Parallel force System:**

This System exists if the lines of action of all the forces are parallel. This system may be Coplanar (or) non-Coplanar.



and the sum of the components in the y-direction is

$$Y = \Sigma F_y = F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + \cdots + F_n \sin\alpha_n \quad (2)$$

where $F_1 \sin\alpha_1, F_2 \sin\alpha_2, \ldots, F_n \sin\alpha_n$ are resolved parts of $F_1, F_2, \ldots F_n$ respectively along OY.

Let R be the resultant of the forces $F_1, F_2, \ldots F_n$ and θ is the angle which R makes with OX.

$\Sigma F_x = R\cos\theta$ and $\Sigma F_y = R\sin\theta$. ($\because$ Algebraic sum of resolved parts of forces $F_1, F_2, \ldots F_n$ in any direction is equal to the resolved part of the resultant R in that direction).

from (1),

$x = F_1 \cos\alpha_1 + F_2 \cos\alpha_2 + \cdots + F_n \cos\alpha_n = R\cos\theta$

from (2)

$Y = F_1 \sin\alpha_1 + F_2 \sin\alpha_2 + \cdots + F_n \sin\alpha_n = R\sin\theta$

i.e. $X = R\cos\theta$, $Y = R\sin\theta$

$\Rightarrow R^2(\cos^2\theta + \sin^2\theta) = X^2 + Y^2$

$\Rightarrow R^2 = X^2 + Y^2$

$\Rightarrow R = \sqrt{X^2 + Y^2}$ which gives the magnitude of the resultant.

Also,

$$\frac{R\sin\theta}{R\cos\theta} = \frac{Y}{X}$$

$\Rightarrow \tan\theta = \frac{Y}{X}$

which gives the direction of the resultant provided $\cos\theta = \frac{X}{R}$ and $\sin\theta = \frac{Y}{R}$ hold

0°, $180^\circ - C$; $(180^\circ - C) + (180^\circ - A)$.

0°, $180^\circ - C$, $360 - (C+A)$

0°, $180^\circ - C$, $-(C+A)$.

The forces being proportional to $\cos A$, $\cos B$, $\cos C$ can be taken as:

$P = k\cos A$, $Q = k\cos B$, $R = k\cos C$ where k is a constant.

Also, let the resultant be S, making an angle θ with the x-axis.

Then, resolving forces along the x-axis, we have

$$S\cos\theta = P\cos 0^\circ + Q\cos(180 - C) + R\cos(-C-A)$$
$$= P - Q\cos C - R\cos B \quad \text{---} (1)$$

$(\because \cos(-C-A) = \cos(C+A) = \cos(180^\circ - B) = -\cos B)$

Also, resolving forces perpendicular to the x-axis, we have

$$S\sin\theta = P\sin 0^\circ + Q\sin(180^\circ - C) + R\sin(-C-A)$$
$$= Q\sin C - R\sin B \quad (2)$$

$(\because \sin(-C-A) = -\sin(C+A) = -\sin(180 - B) = -\sin B)$

Squaring and adding (1) & (2), we get

$$S^2 = P^2 + Q^2(\cos^2 C + \sin^2 C) + R^2(\cos^2 B + \sin^2 B) + 2QR(\cos B\cos C - \sin B\sin C) - 2RP\cos B - 2PQ\cos C$$
$$= P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cos B - 2PQ\cos C.$$

$(\because \cos B\cos C - \sin B\sin C = \cos(B+C) = \cos(180^\circ - A) = -\cos A)$

Replacing P, Q, R by $k\cos A$, $k\cos B$, $k\cos C$; we get

$$S^2 = k^2[\cos^2 A + \cos^2 B + \cos^2 C - 6\cos A\cos B\cos C)$$

But $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$. (By trigonometry)

$\therefore S^2 = k^2(1 - 8\cos A\cos B\cos C)$.

$\Rightarrow S = k\sqrt{1 - 8\cos A\cos B\cos C}$ proportional to $\sqrt{1 - 8\cos A\cos B\cos C}$.

If coplanar forces $F_1, F_2, \ldots F_n$ acting at the point O be in $\rightleftharpoons$ then,

$$\bar{R} = 0$$

$$\therefore |R| = \sqrt{X^2+Y^2} = 0$$

which is true only if $X=0$ and $Y=0$

Hence the conditions are necessary.

Sufficient conditions:

if $X=0$ and $Y=0$ then the forces are in $\rightleftharpoons$.

$\because \bar{R}\cdot\hat{i} = X$ and $\bar{R}\cdot\hat{j} = Y$

if $X=0$ and $Y=0$

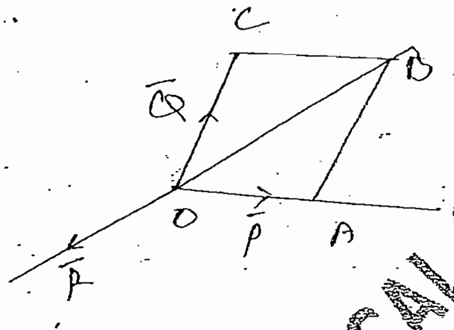
then $\bar{R}\cdot\hat{i} = 0$ & $\bar{R}\cdot\hat{j} = 0$

Since $\hat{i}$ and $\hat{j}$ are not zero and R cannot be $\perp r$ to $\hat{i}$ and $\hat{j}$ as they are coplanar.

$\therefore$ we have $\bar{R} = 0$

Hence the forces are in $\rightleftharpoons$.

or, $\bar{P} + \bar{Q} = -\bar{R}$ —(1)



Let $\bar{OA} = \bar{P}$, $\bar{OC} = \bar{Q}$

Completing the parallelogram OABC, we have,

$$\bar{AB} = \bar{OC} = \bar{Q}$$

and $\bar{OA} + \bar{AB} = \bar{OB}$ —(2)

From (1) & (2), we have,

$$\vec{OB} = -\bar{R}$$

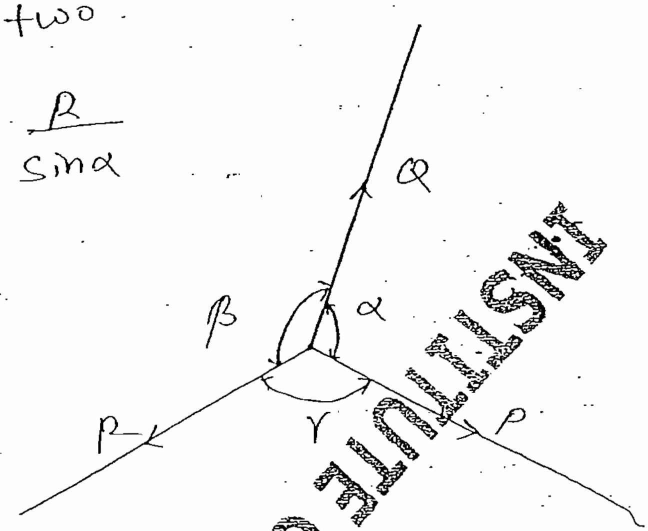
or, $\vec{BO} = \bar{R}$

Thus the sides OA, AB and BO of ΔABC represent the forces $\bar{P}$, $\bar{Q}$, $\bar{R}$ in magnitude and direction, taken in order.

Lami's Theorem

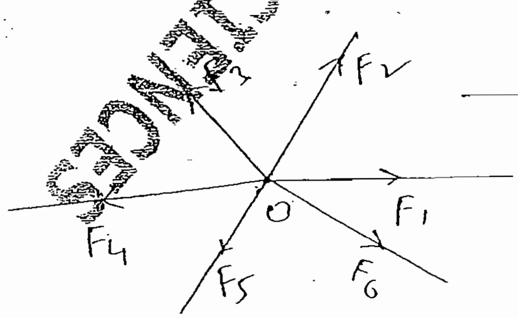
If three forces acting on a particle keep it in $\rightleftharpoons$ each is proportional to the ~~sides~~ sine of the angle between the other two

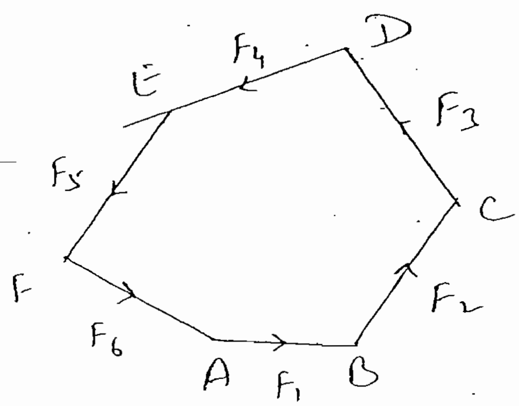
i.e $\dfrac{P}{\sin\beta} = \dfrac{Q}{\sin\gamma} = \dfrac{R}{\sin\alpha}$



Polygon of forces

if any number of forces, acting on a particle be represented, in magnitude and direction, by the sides of a closed polygon, taken in order the forces shall be in $\rightleftharpoons$.





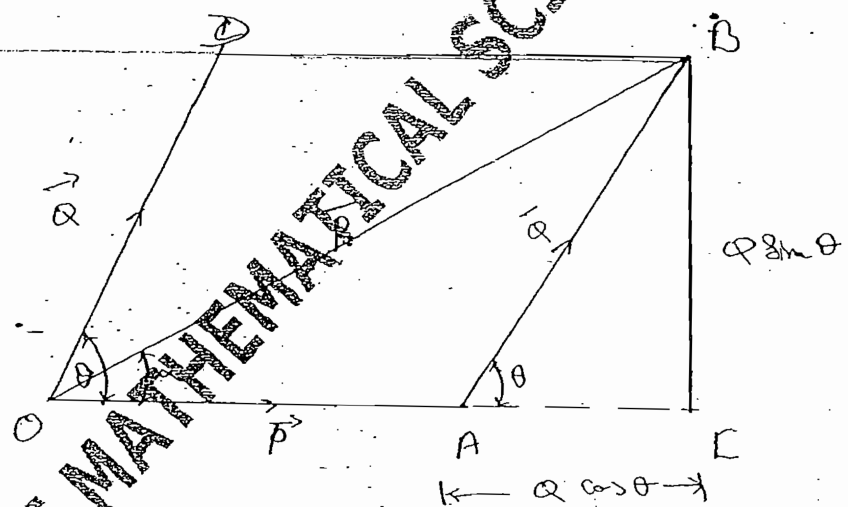
IMS
INSTITUTE OF MATHEMATICAL SCIENCES
Head Off.: 105 – 106, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, Delhi-9.
Branch Off.: 87, First Floor (Back Side), Old Rajender Nagar Market, Delhi-110060.
09999329111, 09999197625

(2)

Q.1. The resultant of forces P and Q is R. if Q is doubled in magnitude, R is doubled in magnitude. if Q is reversed, R is again doubled in magnitude. Show that

$P:Q:R = \sqrt{2}:\sqrt{3}:\sqrt{2}$

Soln:



Force representations are shown in figure.

$\bar{R} = \bar{P} + \bar{Q}$

$R^2 = P^2 + Q^2 + 2PQ\cos\theta$ — (1)

When Q is doubled, R is doubled.

$\therefore (2R)^2 = P^2 + (2Q)^2 + 2P(2Q)\cos\theta$

$\Rightarrow 4R^2 = P^2 + 4Q^2 + 4PQ\cos\theta$ — (2)

when Q is reversed, the resultant $\bar{R}$ is again doubled in magnitude.

$\therefore$ angle between P and Q $= \pi - \theta$

$\therefore \; (2R)^2 = P^2 + (+Q)^2 + 2\cdot P(+Q)\cos(\pi-\theta)$

$\Rightarrow \; 4R^2 = P^2 + Q^2 - 2PQ\cos\theta$ —— (3).

Adding (1) & (3), we have

$5R^2 = 2P^2 + 2Q^2$

$\Rightarrow \; 2P^2 + 2Q^2 - 5R^2 = 0$ —— (4)

From, $2\times(3) + (2)$, we have

$12R^2 = 3P^2 + 6Q^2$

i.e. $3P^2 + 6Q^2 - 12R^2 = 0$ — (5)

From (4) & (5), we have,

$$\frac{P^2}{6} = \frac{Q^2}{9} = \frac{R^2}{6}$$

or, $\dfrac{P^2}{2} = \dfrac{Q^2}{3} = \dfrac{R^2}{2}$

$\therefore P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$

Proved

Q.2. The greatest resultant which two forces can have is P and least is Q. Show that if they act at an angle θ the resultant is of magnitude

$$\left(P^2\cos^2\theta/2 + Q^2\sin^2\theta/2\right)^{1/2}$$

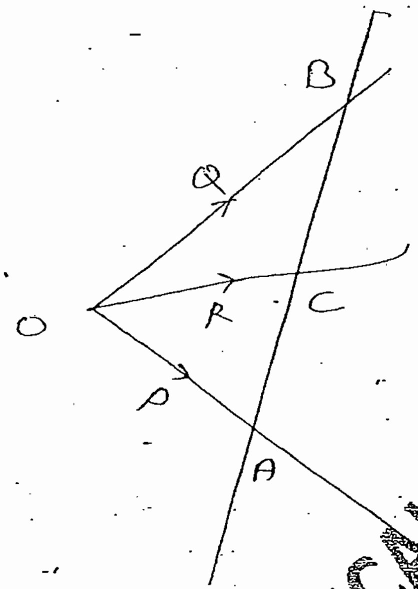
Hints: Let F_1, F_2 be two forces. $F_1 = |\bar{F_1}|$, $F_2 = |\bar{F_2}|$

$\therefore P = |\bar{P}| = |\bar{F_1}| + |\bar{F_2}|$, $Q = |\bar{Q}| = |\bar{F_1}| - |\bar{F_2}|$

$F_1 = \frac{P+Q}{2}$, $F_2 = \frac{P-Q}{2}$

forces P and Q act at O and have a resultant R. if any transversal cuts their line of action at A, B and C respectively then show that

$$\frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$



in vector notation, force P can be written as

$$\vec{P} = P\,\frac{\vec{OA}}{OA}$$

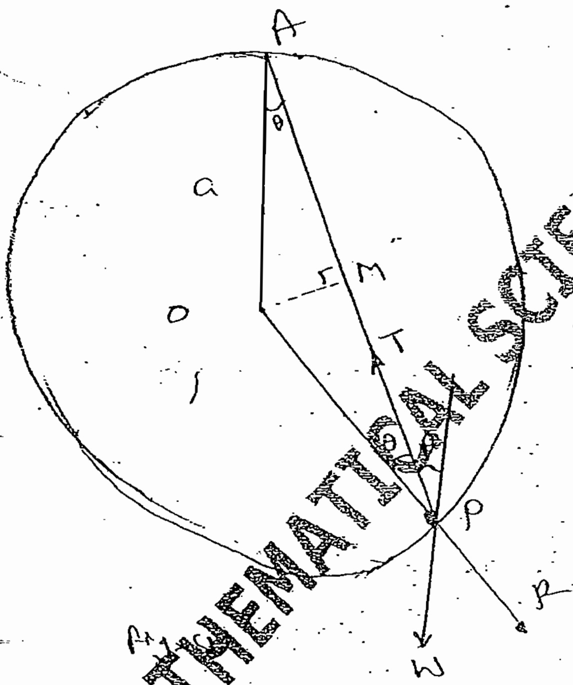
$$= \frac{P}{OA}\,\vec{OA}$$

Similarly $\vec{Q} = \frac{Q}{OB}\,\vec{OB}$

From $\lambda - \mu$ theorem, resultant $\vec{R}$ of two forces $\vec{P}$ and Q is given by

$$\vec{R} = \left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC} \quad — (1)$$

Soln



Let 'O' be the centre of the circular wire of radius 'a'.

One end of the string AP of length 'l' is fixed at A (at the highest point of the wire) and a ring of weight 'W' is attached to the other end of the string, be in ⇌ when it is at the point P of the wire.

Free Body Diagram of the ring is shown below.

But $\vec{R}$ can be written as

$$\vec{R} = \frac{R}{OC}\,\vec{OC} \quad -(2)$$

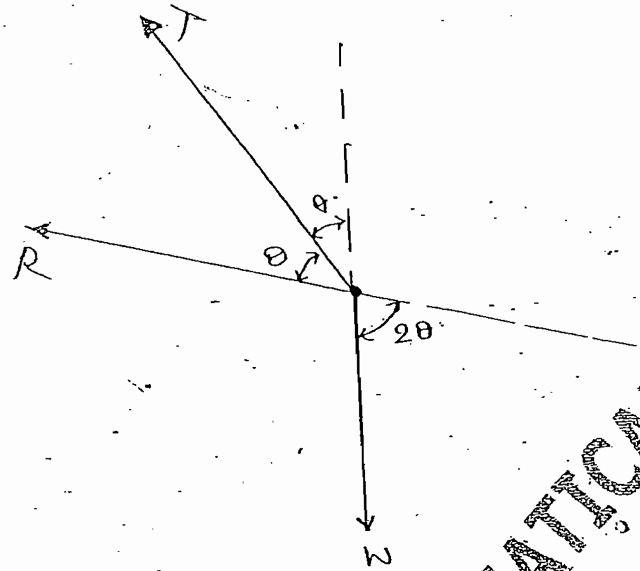
$\therefore$ from (1) & (2), we have;

$$\left(\frac{P}{OA} + \frac{Q}{OB}\right)\vec{OC} = \frac{R}{OC}\,\vec{OC}$$

$$\Rightarrow \frac{P}{OA} + \frac{Q}{OB} = \frac{R}{OC}$$

Proved

Q.4. One end of a light inextensible string of length l is fastened to the highest point of a smooth circular wire of a radius 'a' which is kept fixed in a vertical plane. The other end of the string is attached to a small heavy ring of weight 'W' which slides on the wire. Find the tension of the string and the reaction of the wire.



Where, W = weight of the ring

T = Tension in the string

R = Reaction force on the ring applied by circular wire along the normal PO

Applying lami's theorem, we have,

$$\frac{T}{\sin(\pi-2\theta)} = \frac{W}{\sin\theta} = \frac{R}{\sin(\pi+\theta)}$$

$$\Rightarrow \frac{T}{\sin 2\theta} = \frac{W}{\sin\theta} = \frac{R}{-\sin\theta}$$

$\therefore$ $R = -W$ (negative sign shows, it will act on opp. dirⁿ)

and $T = W \cdot \frac{\sin 2\theta}{\sin\theta}$

Since, $l = AP$ and $OA = a$

$\therefore$ from figure (1)

$$l = 2 \cdot a \cdot \cos\theta$$

$$\therefore \cos\theta = \frac{l}{2a}$$

$\therefore$ Tension in the String

$$T = \frac{Wl}{a}$$

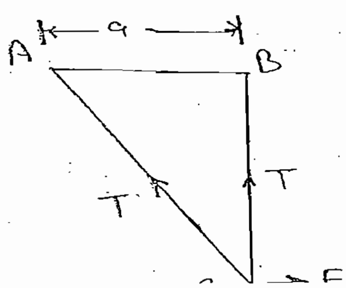
and Reaction of the wire,

$$R = W$$

$\Leftarrow$ Ans

Q.5. A String of length 'l' is fastened to two points A and B at the same level at a distance 'a' apart. A ring of weight 'W' can slide on the String and a horizontal force 'F' is applied to it Such that the ring is in $\rightleftharpoons$ vertically below B. Show that

$$F = W\frac{a}{l} \text{ and that the tension in the string is } \frac{W(l^2+a^2)}{2l^2}$$



Hints: $AC + BC = l$

apply Lami's theorem

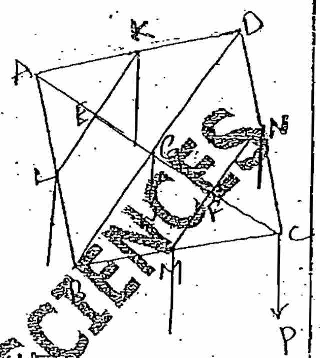
(4) Three forces P, Q, R acting on a particle are in equilibrium. If the angle b/n P and Q is double the angle b/n P and R then prove that

$$P = \left(\frac{R^2 + Q^2}{Q}\right)$$

(5) A and B are two fixed points in a horizontal line at a distance 'c' apart. Two fine light strings AC & BC of length a and b respectively support a mass at C. Show that the tensions of the strings are in the ratio

$$b(a^2 + c^2 - b^2) : a(b^2 + c^2 - a^2)$$

Sol^n: Let ABCD be the top of the table.



Let the legs be at L, M, N, K. The weight of the table acts at G, the centre of gravity of the table

Let the weight P be placed at the corner C.

Taking moments of P and W about the line $\overleftrightarrow{MN}$, we have

$$W \cdot |GF| = P \cdot |FC|$$

But $|GF| = |FC|$

$$\therefore W|FC| = P|FC|$$

$$\Rightarrow P = W$$

Varignon's theorem:

The algebraic sum of the moments of two coplanar forces, not forming a couple, about any point in their plane, is equal to the moment of their resultant about that point.

A force F acts at a point (3,4) of the xy-plane. The force is directed away from the origin and inclined at 60° to the x-axis. The horizontal component of F is 5 kg. wt.

(a) Determine the force F.

(b) Using varignon's Theorem, calculate the moment of F about the origin.

(c) Hence find the perpendicular distance of the origin from the line of action of F.

Draw $AT \perp BC$.

From $\Delta DOL \sim \Delta DAT$,

we have,

$$\frac{AT}{OL} = \frac{AD}{OD} = \frac{3OD}{OD}$$

$$\Rightarrow OL = \frac{AT}{3}.$$

Now if S = area of ΔABC

then, $S = \frac{1}{2} \cdot AT \cdot BC$

$$\Rightarrow AT = \frac{2S}{BC}$$

$$= \frac{2S}{a}$$

$$OL = \frac{2}{3}\frac{S}{a}$$

Similarly, $OM = \frac{2S}{3b}$ & $ON = \frac{2S}{3c}$

Putting the value of OL, OM & ON in eq^n (1),

we have,

$$P \cdot \frac{2S}{a} + Q \cdot \frac{2S}{3b} + R \cdot \frac{2S}{3c} = 0$$

$$\Rightarrow \frac{P}{a} + \frac{Q}{b} + \frac{R}{c} = 0$$

$$\vec{M} = \vec{OA_r} \times \vec{F_r}$$
$$= \vec{a_r} \times \vec{F_r}$$

and besides this couple a single force F_r ∥ to OP is left at O.

Thus the single force F_r acting at A_r is equivalent to a force F_r acting at 'O' and a couple of moment $\vec{a_r} \times \vec{F_r}$.

Similarly all n forces $\bar{F_1}, \bar{F_2}, \bar{F_3}, \ldots \bar{F_n}$ acting at the point $A_1, A_2, \ldots A_n$ whose position vectors refferred to O are

$$\bar{a_1}, \bar{a_2}, \ldots \bar{a_n}$$

respectively are equivalent to single forces $F_1, F_2, \ldots F_n$ acting at O and couples of moments

$$\bar{a_1} \times \bar{F_1}, \bar{a_2} \times \bar{F_2}, \ldots a_n \times \bar{F_n}$$

Hence, the given system of forces will be equivalent to forces $F_1, F_2, F_3, \ldots F_n$ acting at O, together with couples of moments

$$\bar{a_1} \times \bar{F}, \bar{a_2} \times \bar{F_2}, \ldots \bar{a_n} \times \bar{F_n}$$

If $\bar{R}$ is the resultant of the concurrent forces $\bar{F_1}, \bar{F_2}, \bar{F_3}, \ldots \bar{F_n}$ acting at 'O', then, we have

$$\bar{R} = \bar{F_1} + \bar{F_2} + \bar{F_3} + \cdots + \bar{F_n} = \sum_{r=1}^{n} F_r$$

and if M is the moment of the resultant couple of the above couples, then

$$M_1 = \bar{a}_1 \times \bar{F_1} + \bar{a}_2 \times \bar{F_2} + \cdots + F_n$$

$$= \sum_{r=1}^{n} a_r \times F_r$$

Hence the system of coplanar forces acting on a body is equivalent to a single force

$$\bar{R} = \sum_{r=1}^{n} F_r$$

acting at an arbitrarily chosen point 'O' together with a couple of moment

$$\bar{M} = \sum_{r=1}^{n} \bar{a}_r \times \bar{F_r}$$

* Necessary and sufficient conditions for $\rightleftharpoons$ of a rigid body.

The necessary and sufficient conditions for the $\rightleftharpoons$ of a rigid body under the action of a system of coplanar ~~forces~~ acting at different points of it are that the sums of the resolved parts of the forces in any two mutually $\perp$r directions vanish seperately and the sum of the moments of the forces about any point in the plane ...

Cartesian form : From (1), we have,

$$xi + yj = \left(\frac{M}{R_y}\right)i + t(R_x i + R_y j)$$

Equating the coefficients of i and j on both sides, we have,

$$x = \frac{M}{R_y} + tR_x, \quad y = tR_y$$

Eliminating 't' from these eqn, we have,

$$x = \frac{M}{R_y} + \frac{y}{R_y} \cdot R_x$$

$$\Rightarrow \boxed{R_y \cdot x - R_x \cdot y = M}$$

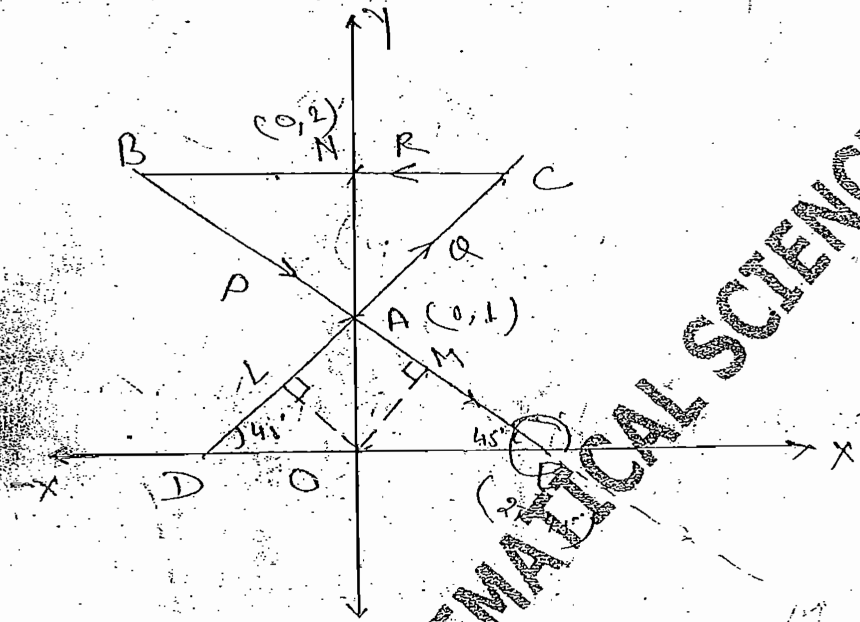
Which is the equation of the line of action of the resultant in Cartesian form.

Q4. Three forces P, Q, R act along the sides of a Δ formed by the line $x+y=1$, $y-x=1$ and $y=2$. Find the equation of the line of action of the resultant.

Soln. Let the three forces P, Q, R act along the lines

$x+y=1$, $y-x=1$ and $y=2$ i.e. the lines RP

$\angle A(90)$ (B) ... perpendicular



Let $\vec{F}$ be the resultant force of $\bar{P}, \bar{Q}$ & $\bar{R}$

$\therefore \vec{F} = \vec{P} + \vec{Q} + \vec{R}$

if F_x = resultant force in x-direction

then $F_x = P\cos(2\pi - 45°) + Q\cos(45°) + R\cos(\pi)$

$= \frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} - R$

$\Rightarrow F_x = \frac{1}{\sqrt{2}}(P + Q - \sqrt{2}R)$

& if F_y = resultant force in Y-direction

then,

$$F_y = P\sin(2\pi - 45^\circ) + Q\sin 45^\circ + R\sin\pi$$

$$= -\frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} + 0$$

$$\Rightarrow F_y = \frac{1}{\sqrt{2}}(Q-P)$$

Let M be the algebric sum of the moments of the forces about the origin 'O'

$$\therefore M = -P\cdot OM - Q\cdot OL + R\cdot ON$$

$$= -P\cdot OA\sin 45^\circ - Q\cdot OA\sin 45^\circ + R\cdot 2$$

$$= -\frac{1}{\sqrt{2}}(P+Q) + 2R$$

$\therefore$ the equation of the line of action of resultant force

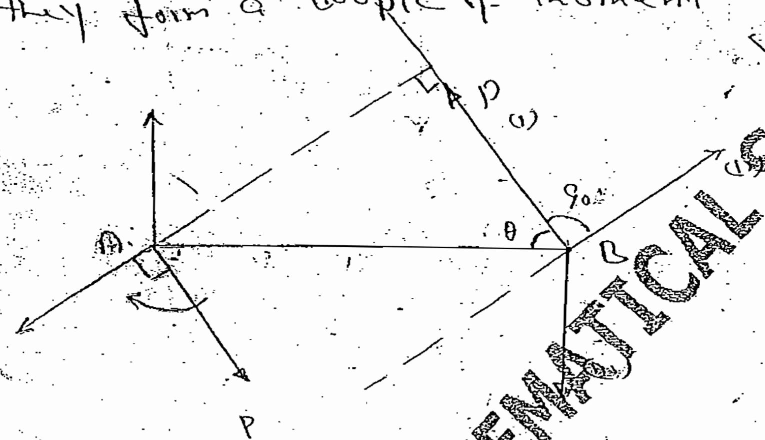
$$F_y\cdot x - F_x\cdot y = M$$

$$\Rightarrow \frac{1}{\sqrt{2}}(Q-P)x - \frac{1}{\sqrt{2}}(P+Q-\sqrt{2}R)y = -\frac{1}{\sqrt{2}}(P+\ldots) + 2\ldots$$

$$\Rightarrow (Q-P)x - (P+Q-\sqrt{2}R)y = 2\sqrt{2}R - (\ldots$$

Ans

Q. 2. Two equal unlike || forces acting at fixed points A and B form a couple of moment G; if their lines of action are turned through one right angle they form a couple of moment H. Show that when they both act at right angle to AB, they form a couple of moment $\sqrt{G^2+H^2}$

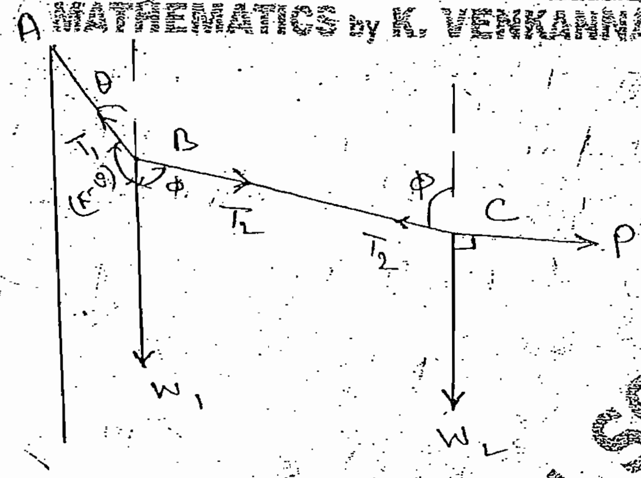


Q. 3. Weights W_1, W_2 are fastened to a light inextensible string ABC at the point B, C, the end A being fixed. Prove that, if a horizontal force 'P' is applied at 'C' and in equilibrium AB and BC are inclined at angle θ, φ to the vertical, then

$$P = (W_1+W_2)\tan\theta = W_2\tan\phi$$

Soln. Forces are shown in the figure.

Let
T_1 = tension in the string AB
T_2 = tension in the string BC



Now, applying Lami's theorem at point 'C'

we have,

$$\frac{P}{\sin(\pi-\phi)} = \frac{T_2}{\sin 90^\circ} = \frac{W_2}{\sin(\pi/2+\phi)}$$

$$\Rightarrow \frac{P}{\sin\phi} = \frac{T_2}{1} = \frac{W_2}{\cos\phi}$$

$$\therefore P = W_2 \tan\phi \quad \& \quad T_2 = \frac{W_2}{\cos\phi} \quad \text{--- (1)}$$

Again applying the Lami's theorem at B,

$$\frac{T_1}{\sin\phi} = \frac{T_2}{\sin(\pi-\theta)} = \frac{W_1}{\sin(\pi-\phi+\theta)}$$

$$\Rightarrow \frac{T_1}{\sin\phi} = \frac{T_2}{\sin\theta} = \frac{W_1}{\sin(\phi-\theta)}$$

$$\Rightarrow \frac{T_2}{\sin\theta} = \frac{W_1}{\sin(\phi-\theta)}$$

$$\text{or, } \frac{W_2}{\cos\phi\cdot\sin\theta} = \frac{W_1}{\sin(\phi-\theta)} \quad [\text{from (1)}]$$

$$W_2 = \frac{W_1 \cos\phi \cdot \sin\theta}{\sin\phi \cdot \cos\theta - \cos\phi \cdot \sin\theta}$$

$$\Rightarrow W_2 = \frac{W_1}{\tan\phi \cdot \cot\theta - 1}$$

$$\Rightarrow W_2(\tan\phi \cdot \cot\theta - 1) = W_1$$

$$\Rightarrow (W_2 \tan\phi) \cot\theta - W_2 = W_1$$

$$\Rightarrow P = (W_1 + W_2)\tan\theta \quad \text{--- (2)}$$

[from (1), $P = W_2 \tan\phi$]

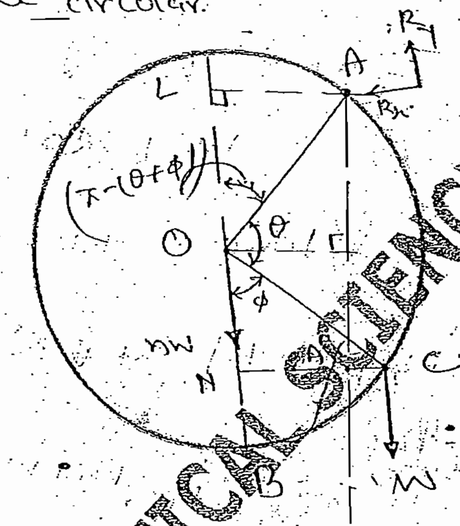
$\therefore$ from (1) & (2),

$$P = (W_1 + W_2)\tan\theta = W_2 \tan\phi \quad \underline{\text{Proved}}$$

Q4. A uniform circular disc of weight nW has a heavy particle of weight W attached to a point 'C' on its rim. If the disc is suspended from a point 'A' on its rim, 'B' is the lowest point; and if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of the disc is $2\sec^{-1}(2(n+1))$

Let O = centre of the circular disc.

Case I:



Forces acting on the body is shown in figure.

Let
$\angle AOC = \theta$
& $\angle BOC = \phi$

in this case body is in ___ under the action of three forces nW, W and Reaction at A.

$\therefore$ Total moment about 'A' = 0

i.e. $M_A = 0$

$\Rightarrow nW \cdot AL - W \cdot A'C = 0$

$\Rightarrow n \cdot AL = A'C = NC - NA'$

$\Rightarrow n \cdot AL = NC - LA$

$\Rightarrow (n+1)\, AL = NC$

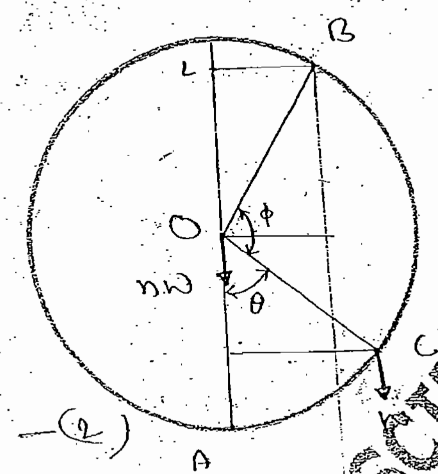
Let r = radius of disc

$\therefore (n+1)\, r \cdot \sin(\pi-(\theta+\phi)) = r\sin\phi$

$\Rightarrow (n+1)\sin(\theta+\phi) = \sin\phi$ —(i)

Case II :

Similarly,



$\sum M_B = 0$

$\Rightarrow (n+1)\sin(\theta+\phi) = \sin\theta$ — (2)

From (1) & (2), we have,

$\theta = \phi$

putting $\theta = \phi$, in (1), we have,

$(n+1)\sin 2\theta = \sin\theta$

$\Rightarrow (n+1)\, 2 \cdot \cancel{\sin\theta} \cdot \cos\theta = \cancel{\sin\theta}$

$\Rightarrow 2(n+1)\cos\theta = 1 \Rightarrow \theta$

$= \sec^{-1}(2(n+1))$

the required angle

$\angle AOB = \theta+\phi = \theta+\theta = 2\theta$

$= 2 \cdot \sec^{-1}(2(n+1))$ Proved

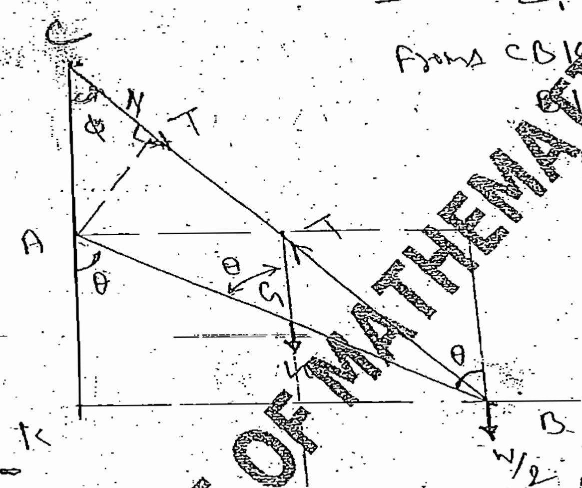
Q5. A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $\frac{W}{2}$, the weight of the rod being W, this end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is $\frac{lW}{c}$.

Hint: $\sum M_A = 0$

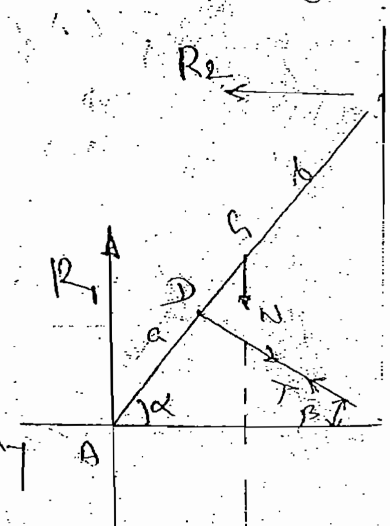
From CBK, ADK

$BK = BK$

length of rod $= 2a$ (say)



Q6. A beam of weight W is divided by its centre of gravity G into two portions AG and GB, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AC and against a smooth wall CB. A string is attached to a ~~hook~~ at C and to the beam at point D. If T be the tension of the string and α & β be the inclinations of the beam and string respectively to the horizon, show that

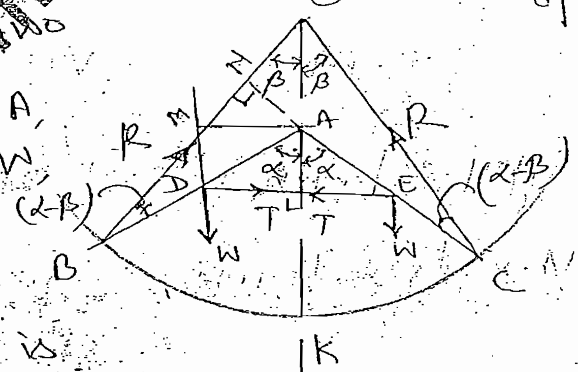


Q: Two equal uniform rods AB, AC each of weight W are freely joined at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being joined by a light string, show that if the string is stretched, its tension is $W(\tan\alpha - 2\tan\beta)$ where, 2α is the angle b/w the rods, and β the angle either rod subtends at the centre.

$(r > l)$ length of rod

Sol:

Let AB, AC be the two rods, freely joined at A, and each having weight W, be placed inside the smooth circular hoop of centre 'O' as shown in figure.



Let the middle point D & E of respective rod be joined by a string, and tension T is induced.

Let R = reaction at B & C by the circular hoop.

Given that $\angle BAK = \alpha = \angle CAK$

and $\angle BOK = \angle COK = \beta$

$\therefore \angle ABO = \angle ACO = \alpha - \beta$

For $\rightleftharpoons$ rod AB and AC

$\Sigma F_y = 0$

$\Rightarrow \quad 2R\cos\beta = W \Rightarrow R = \frac{W}{\cos\beta}$ (1)

Now consider the $\rightleftharpoons$ rods, along A

$\Sigma M_A = 0$

$\Rightarrow \quad R \cdot AN - W \cdot AM - T \cdot AL = 0$

$\Rightarrow \quad R \cdot AN = W \cdot AM + T \cdot AL$

If l = length of each rod

then from figure,

$R l \sin(\alpha-\beta) = W \cdot \frac{l}{2}\sin\alpha + T \cdot \frac{l}{2}\cos\alpha$

put $R = \frac{W}{\cos\beta}$ from (1),

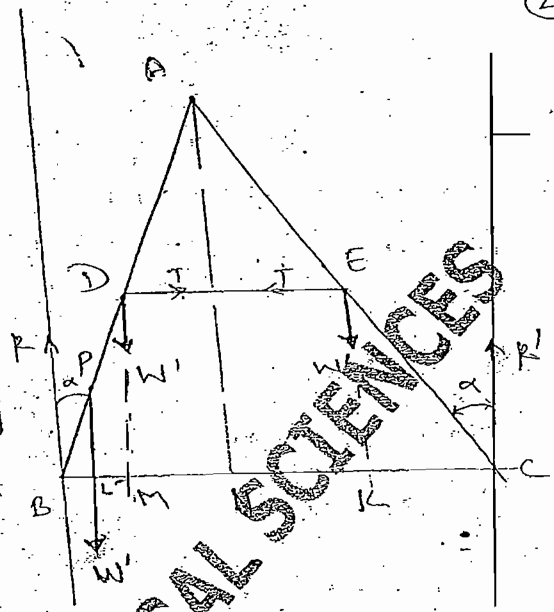
$\frac{W l}{\cos\beta} \cdot \sin(\alpha-\beta) = \frac{W}{2} \cdot \sin\alpha + \frac{T \cdot \cos\alpha}{2}$

$T \cdot \cos\alpha = \frac{2W}{\cos\beta}\left(\sin\alpha \cdot \cos\beta - \sin\beta \cdot \cos\alpha\right) - W \sin\alpha$

$\Rightarrow \quad T \cos\alpha = W\{\sin\alpha - 2 \cdot \tan\beta \cdot \cos\alpha\}$

$\therefore \quad T = W(\tan\alpha - 2\tan\beta)$ Proved

Q. A step ladder in the form of the letter A with each of its legs inclined at an angle 'α' to the vertical is placed on a horizontal floor and is held up by a cord connecting the middle points of its legs, there being no friction anywhere. Show that when a weight W is placed on one of the steps at a height from the floor equal to $\frac{1}{n}$ of the ladder, the increase in the tension of the cord is ($\frac{1}{n}$ W $\tan\alpha$).



Hints: $PL = \frac{1}{n} AN$ $BL = \frac{1}{n} BN = \frac{1}{n} \cdot \frac{BC}{2}$

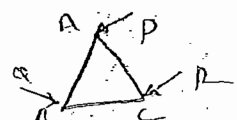
$BM = MN = NK = \dots$ $\therefore BM = \frac{1}{4} BC$, $BK = \frac{3}{4} BC$

$\Sigma M_B = 0 \Rightarrow R' = \frac{1}{2n} W + W'$

For leg, $\Sigma M_A = 0$

* If three forces acting upon a rigid body keep it in equilibrium, they must either meet in a point or be parallel.

$\therefore \bar{P} + \bar{Q} + \bar{R} = 0$ (for $\rightleftarrows$ force)



Also the sum of the moments of the forces about any point must be zero.

$\therefore M_A = 0$

$\vec{AB} \times \bar{Q} + \vec{AC} \times \bar{R} = 0$

$\therefore \vec{AB} \times \bar{Q} = \vec{CA} \times \bar{R}$ ...

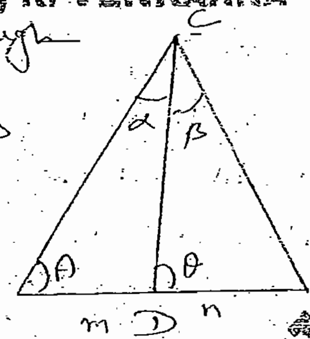
i.e. n vs ... to the plane ...

Since the forces $\bar{Q}$ & $\bar{R}$ are coplanar force, hence their line of action must either intersect or be parallel.

(i) If $\bar{Q}, \bar{R}$ intersect, then force passes through point ... the resultant ...

(ii) If $\bar{Q} \parallel \bar{R}$ ...

* If a line CD be drawn through the vertex C of a ΔABC meeting the opposite side AB in D and dividing it into two parts m and n and the angle 'C' into two parts α and β, and if $\angle CBD = \theta$, then

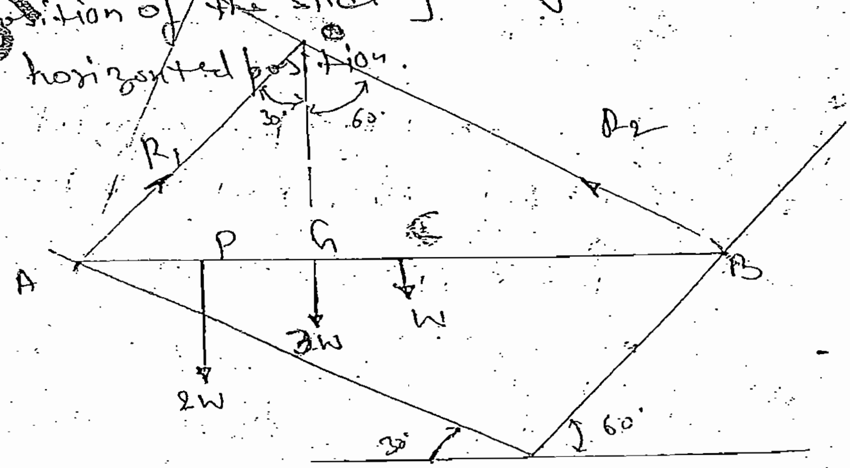


(i) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(ii) $(m+n)\cot\theta = n\cot A - m\cot B$

Q.9. A uniform beam rests with its ends on two smooth inclined planes which makes angles of 30° and 60° with the horizontal respectively. A weight equal to twice that of the beam can slide along its length. Find the position of the sliding weight when the beam rests in a horizontal position.

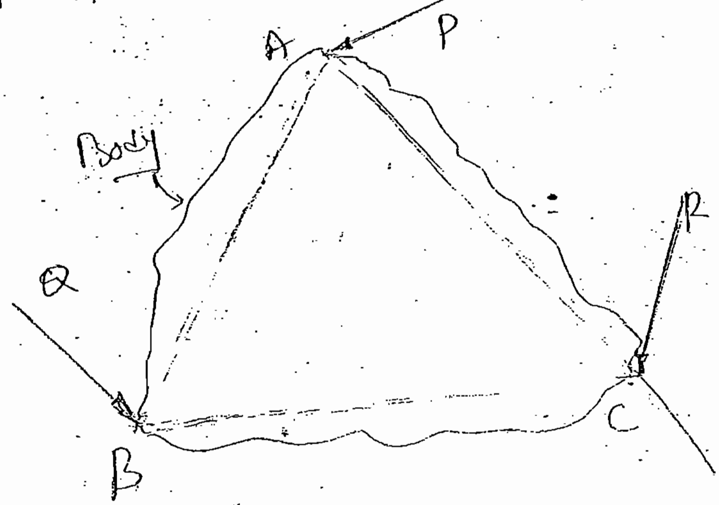
Sol:



Let the beam AB rests in a horizontal position with its ends A & B on two smooth inclined planes ... as shown in figure

*** Equilibrium of a rigid body under the action of three forces only.

<u>theorem</u> If three forces acting upon a rigid body keep it in $\rightleftharpoons$ they must either meet in a point or be parallel.



<u>Proof</u>: let the three forces $\bar{P}, \bar{Q}, \bar{R}$ acting at the points A, B, C respectively of a rigid body, keep it in $\rightleftharpoons$

Since the forces are in $\rightleftharpoons$

$$\therefore \quad \bar{P} + \bar{Q} + \bar{R} = 0 \quad \text{--- } (1)$$

Also the sum of the moments of the forces about any point must be zero.

$$\therefore \quad \Sigma M_A = 0$$

$$\Rightarrow \overrightarrow{AB} \times \bar{Q} + \overline{AC} \times \bar{R} = 0$$

$$\Rightarrow \overrightarrow{AB} \times \bar{Q} = \overrightarrow{CA} \times \bar{R} = \bar{n} \text{ (say)}$$

Thus $\bar{n}$ is a vector $\perp$r to $\overline{AB}$, $\bar{Q}$, $\overline{CA}$ and $\bar{R}$ i.e. vector $\bar{n}$ is $\perp$r to the plane ABC and the $\bar{Q}$ & $\bar{R}$

the forces $\overline{Q}$ and $\overline{R}$ must act in the plane ABC.

$\because$ $\overline{Q}$ & $\overline{R}$ are coplanar, hence their lines of action must either intersect or be parallel.

I: if the $\overline{Q}$ & $\overline{R}$ intersect, then for $\rightleftharpoons$, their resultant must be equal and opposite to the third force $\overline{P}$. Thus the third force $\overline{P}$ must also act in the plane ABC and passes through the point of intersection of the forces $\overline{Q}$ and $\overline{R}$. Hence the three forces are coplanar and concurrent.

II: if the forces $\overline{Q}$ and $\overline{R}$ are parallel then their resultant is also parallel to them and for $\rightleftharpoons$ it must be equal and opposite to the third force $\overline{P}$. Hence the three forces are coplanar and parallel.

Let the sliding weight (2W) be at P in ⇌ position.

R_1, R_2 = reactions at A & B by the inclined plane on the beam AB, respectively.

For ⇌ the vertical line of action of the resultant force (weight) 3W at G will also pass through 'O' as shown in fig.

in Δ AOG,

$AG = OG \tan 30^\circ = \frac{OG}{\sqrt{3}}$

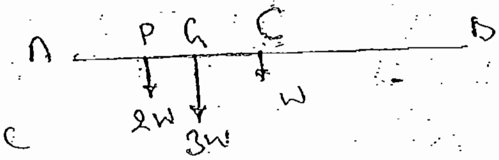
and in Δ BOG, $BG = OG \tan 60^\circ = OG\sqrt{3}$

$\therefore \frac{BG}{AG} = 3 \qquad \frac{BG}{AG} + 1 = 3 + 1$

$\Rightarrow \frac{BG + AG}{AG} = 4$

$\Rightarrow \frac{AB}{AG} = 4 \qquad \Rightarrow AG = \frac{1}{4} AB \quad \text{—(i)}$

But resultant of the weight W at C and 2W at P at G. So for



$\therefore \sum M_A = 0$

$\Rightarrow (3W) AG = 2W \cdot AP + W \cdot AC$

$AG = \frac{1}{3}(2AP + AC$

$$AG = \frac{1}{3}\left(2AP + \frac{AB}{2}\right) \quad \text{--- (2)}$$

$\Rightarrow$ from (1) & (2), we have

$$\frac{1}{4}AB = \frac{1}{3}\left(2AP + \frac{AB}{2}\right)$$

$$\Rightarrow AP = \frac{AB}{8}$$

$$\therefore BP = AB - \frac{AB}{8} = \frac{7}{8}AB$$

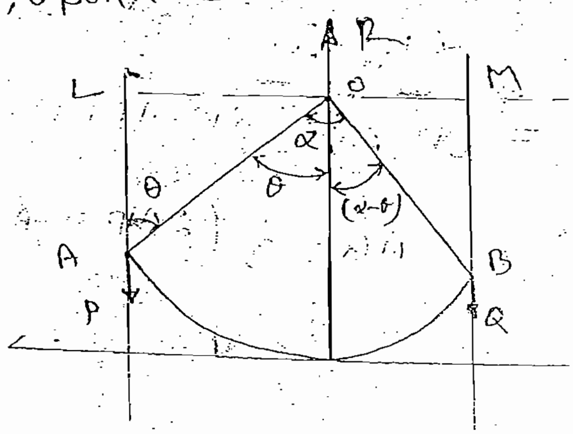
$$\therefore \frac{AP}{BP} = \frac{1}{7}$$

Hence the beam will be in $\rightleftharpoons$ in the horizontal position on the inclined planes when the sliding weight is at the point which divides the beam in the ratio 1 : 7.

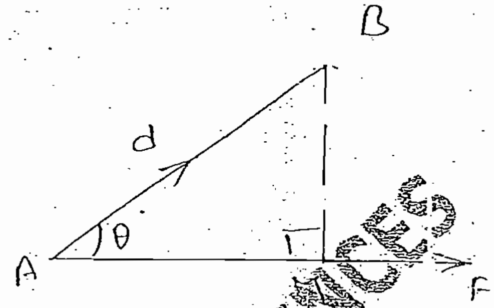
Ans

(10) A rigid wire, without weight, in the form of the arc of a circle subtending angle α at its centre and having weights P and Q at its extermities, rests with its concavity downwards, upon a smooth horizontal plane. Show that, if θ be the inclination to the vertical of the radius to the end at which P is suspended then

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$



Work done by a force



Let a force represented by the vector $\vec{F}$ acts at the point A.

Let the point A be displaced to the point B, where, $\overrightarrow{AB} = \vec{d}$

Then the work done W by a force $\vec{F}$ during the displacement $\vec{d}$ of its point of application is defined as — (i)

$$W = \vec{F} \cdot \vec{d}$$

i.e. W = Scalar product of $\vec{F}$ and $\vec{d}$

Let θ be the angle between the vector $\vec{F}$ & $\vec{d}$

and $F = |\vec{F}|$, $d = |\vec{d}| = AD$

$$\therefore W = F\, d \cos\theta \quad \text{—— (2)}$$

= F × (displacement of the point of application of the $\vec{F}$ in the direction of the force)

Hence, the work done by a force is equal to the magnitude of the force multiplied by the displacement

of the point of application of the force in the direction of the force.

From eq^n (2), we have,

(i) if $\theta = \frac{\pi}{2}$, i.e. if the displacement of the point of application of the force is $\perp_r$ to the direction of the force, then

$$W = 0.$$

(ii) if $0 \le \theta \le \pi/2$ i.e. if the displacement of the point of application of the force parallel to the line of action of the force in the direction of the force, then W is positive.

(iii) if $\frac{\pi}{2} < \theta \le \pi$ i.e. if the displacement of the point of application of the force, ∥el to the line of action of the force is opposite to the direction of the force then W is negative.

* The work done by force $\vec{F}$ acting at the point of $\vec{r}$ during small displacement $\vec{dr}$ of its point of application is

$$\vec{F} \cdot \vec{dr}$$

Work done by a system of concurrent forces

The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the seperate forces.

Proof: Let there be n forces represented by the vectors $\bar{F}_1, \bar{F}_2, \ldots \bar{F}_n$ acting at a point P. Then taking any displacement of P represented by the vector $\bar{d}$, the works done by the seperate forces are respectively equal to

$$\bar{F}_1 \cdot \bar{d}, \ \bar{F}_2 \cdot \bar{d}, \ \ldots \ \bar{F}_n \cdot \bar{d}$$

$\therefore$ Total work done

$$W = (\bar{F}_1 \cdot \bar{d} + \bar{F}_2 \cdot \bar{d} + \cdots \ \bar{F}_n \cdot \bar{d})$$

$$= (\bar{F}_1 + \bar{F}_2 + \cdots + \bar{F}_n) \cdot \bar{d}$$

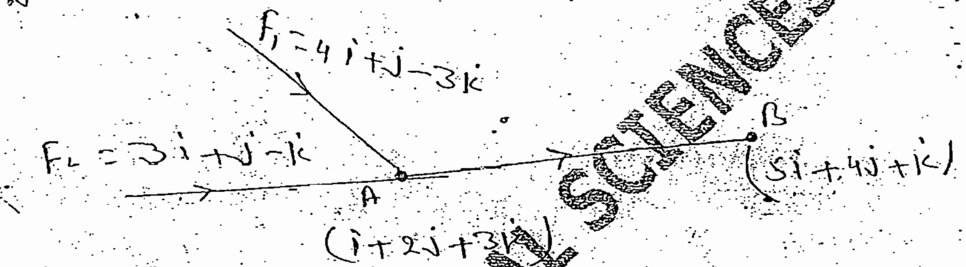
$$= \bar{R} \cdot \bar{d}$$

where, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \cdots + \bar{F}_n$

But $\bar{R} \cdot \bar{d}$ is the work done by the resultant $\bar{R}$ during the displacement $\bar{d}$ of the point P.

Q. A Particle acted on by constant forces $4i + j - 3k$ and $3i + j - k$ is displaced from the point $i + 2j + 3k$ to the point $5i + 4j + k$. Find the total work done by the force.

Sol^n:



Let R = resultant of two forces

d = displacement of particle from A to B.

$\therefore \vec{R} = \vec{F_1} + \vec{F_2}$

$= (4i + j - 3k) + (3i + j - k)$

$= 7i + 2j - 4k$

& $\vec{d} = (5i + 4j + k) - (i + 2j + 3k)$

$= 4i + 2j - 2k$

$\therefore$ Total work done

$W = \vec{F} \cdot \vec{d}$

$= (7i + 2j - 4k) \cdot (4i + 2j - 2k)$

$= 28 + 4 + 8$

$= 40$ units of work Ans

Principal of Virtual Work

Virtual displacement (δs): of a point is any arbitrary infinitesimal change in the position of the point consistent with the constraints imposed on the motion of the point. This displacement can be imagined.

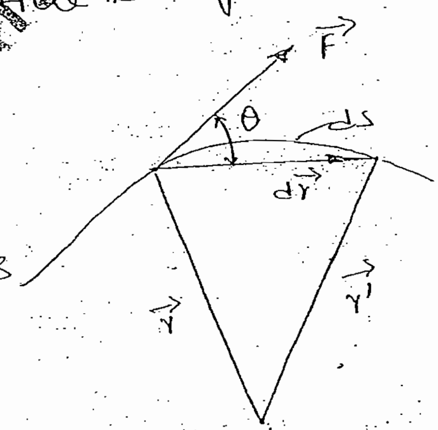
Work of a force and work of a couple moment

→ work of a force on a particle is defined as

$$W = \vec{F} \cdot \vec{dr}$$

or, $W = F\, ds \cos\theta$

or, $W = F\, ds \cos\theta$



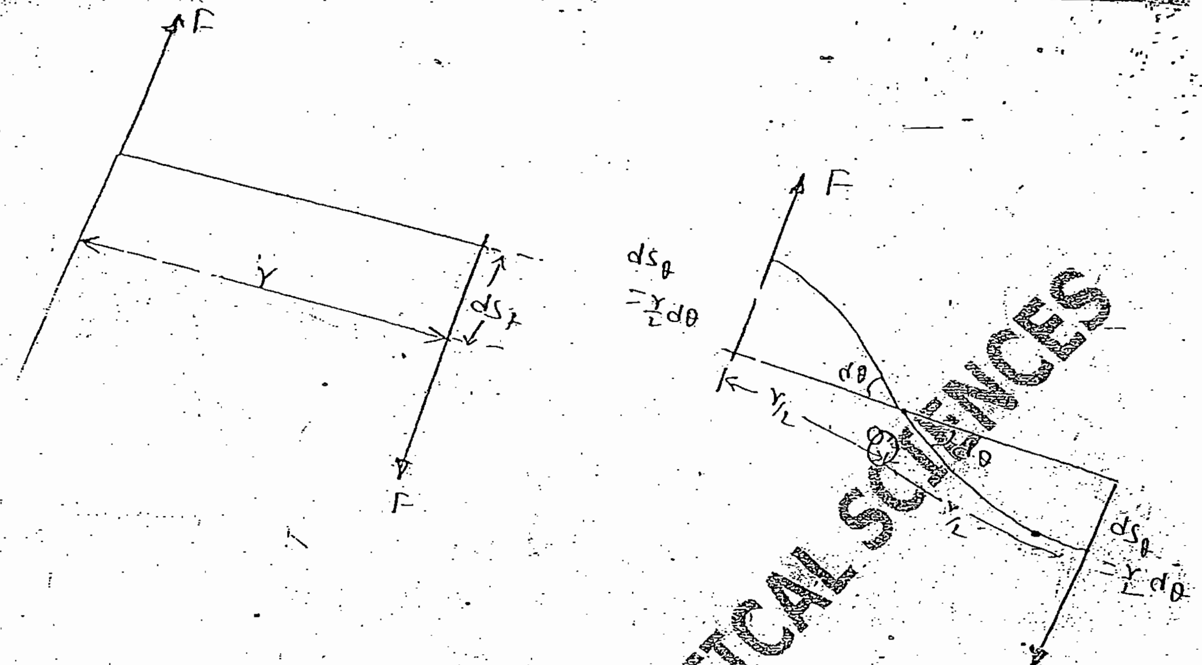
→ Any general differential displacement of a body can be considered as a combination of translation and rotation.

→ when a body is subjected to couple, the work corresponds to translation is zero. While the work corresponds to rotation is

$$dW = \vec{F} \cdot \left(\frac{r}{2} d\theta\right) + \vec{F} \cdot \left(\frac{r}{2} d\theta\right) = F\, r\, d\theta$$

$$= M d\theta$$

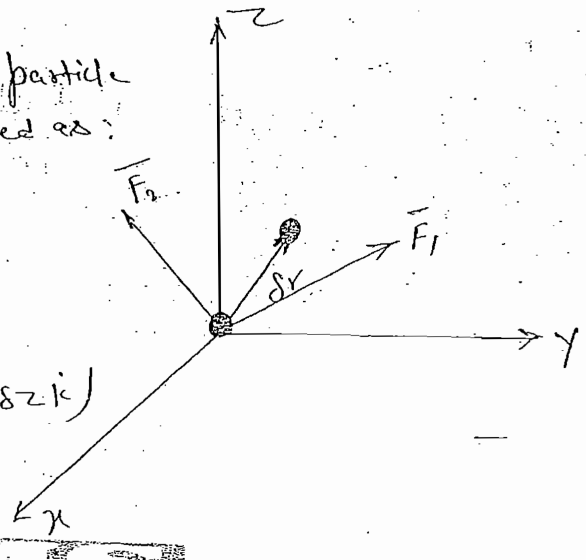
or, $dW = \vec{M} \cdot \vec{d\theta}$



→ For a body under static equilibrium, the virtual work is defined by external forces multiplying the 'virtual' movement along the direction of external forces, i.e.

$$\delta W = \vec{F} \cdot \vec{\delta r} \quad \text{or,} \quad \delta W = \vec{M} \cdot \vec{\delta \theta}$$

The virtual work for a particle under can be expressed as:



$$\delta W = \Sigma F \cdot \delta r$$

$$= (\Sigma F_x i + \Sigma F_y j + \Sigma F_z k) \cdot (\delta x i + \delta y j + \delta z k)$$

$$= \Sigma F_x \delta x + \Sigma F_y \delta y + \Sigma F_z \delta z = 0$$

Principle of virtual work

the necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces is in ⇌ is that the algebric sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation.

Proof Let a system of forces $F_1, F_2, \ldots F_n$ act at the points of rigid body whose position vectors w.r.t some origin O are $\vec{r_1}, \vec{r_2}, \ldots \vec{r_n}$.

Suppose this system of forces is equivalent to a single force, $\vec{R} = \sum F_i$ acting at O, together with a couple of moment $\vec{M} = \sum \vec{r_i} \times \vec{F_i}$.

Then during any small displacement of the body consisting of uniform translation $\vec{U}$ and a small rotation $\vec{\delta\theta}$ about O,

∴ the sum of the works done by these forces

$$W = \vec{U} \cdot \vec{R} + \vec{\delta\theta} \cdot \vec{M} \quad \text{—(1)}$$

The conditions necessary: Let the given system be in ⇌. then $\vec{R} = 0$ and $\vec{M} = 0$

∴ From (1), the sum of the works done by the forces is zero. Hence the condition is necessary.

The condition [illegible] sum of the works done by the forces during any small displacement is zero. Then to prove that the forces are in $\rightleftharpoons$

$$\vec{u}\cdot\vec{R} + \vec{\delta\theta}\cdot\vec{G} = 0 \quad -(2)$$

for any small displacement consisting of a uniform translation $\vec{u}$ and a small rotation $\vec{\delta\theta}$ ...

if $\delta\theta = 0$ and $\vec{u} \neq 0$

then, $\vec{u}\cdot\vec{R} = 0$

also if $\vec{u} \not\perp \vec{R}$

$\therefore \vec{R} = 0$

Now taking $\delta\theta \neq 0$ and $\vec{u} = 0$

$\therefore \vec{\delta\theta}\cdot\vec{G} = 0$

if $\vec{G}$ is not $\perp$r to $\vec{\delta\theta}$ then

$\vec{G} = 0$

for any small displacement $\delta\theta$ and $\vec{u}$,

we must have $\vec{R} = 0$ and $\vec{G} = 0$

* The equation (2) formed by the equating to zero the sum of the virtual works done by the forces is called the equation of virtual work.

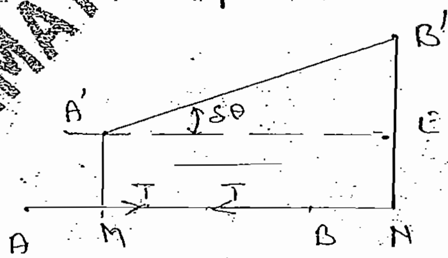
* The above principle of virtual work and its proof equally holds whether the forces are coplanar or not and whether the forces act upon a particle or upon a rigid body

* Forces which are omitted in forming the equation of virtual work

The principle of virtual work gives us a very powerful method of tackling the problems on equilibrium. The mechanical advantage of this principle over other methods is that there are certain forces which are omitted in forming the equation of virtual work; consequently the solution of the problem becomes easy by this method.

(i) The work done by the tension of an inextensible string is zero during a small displacement.

Proof: Let AB be a inextensible string of length l.



T = tension in the string AB.

After a small displacement let the new position of AB is A'B' $\therefore$ AB = A'B'

$\therefore$ The work done by the tension of string AB, during the displacement

$$W = T \cdot AM - T \cdot BN$$
$$= T(AB - MB) - T(MN - MB)$$
$$= T(AB - MN)$$
$$= T \cdot (AB - A'E)$$
$$= T \cdot (AB - A'B' \cos\delta\theta) \quad [\because AB = A'B' = l]$$
$$= T \cdot l\,(1 - \cos\delta\theta)$$
$$= T \cdot l\left\{1 - \left(1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^4}{4!} - \cdots\right)\right\}$$

$w = T \cdot 1 \cdot 0 \quad [\because (\delta\theta)^2 \approx 0]$

$= 0$

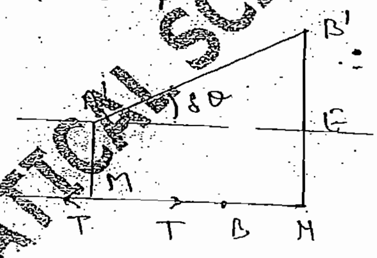
Hence the work done by the tension of an inextensible string is zero during a small displacement.

(ii) The work done by the thrust of an inextensible rod is zero during a small displacement.

Similarly as above,

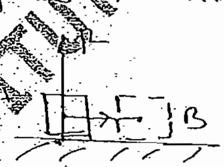
$w = -T \cdot AM + T \cdot BM$

$= 0$



* from (i) and (ii) we can say that if the distance between two particles of a system is invariable, the work done the mutual action and reaction between the two particles is zero.

(iii) The reaction $\vec{R}$ of any smooth surface with which the body is in contact does no work.



$\Rightarrow \because \vec{R} \perp \vec{AB}$ for small displacement

$\therefore w = 0$

But for rough surface

work done by the friction force $= -f_r \cdot AB$

= virtual work

(iv) If a body rolls without sliding on any fixed surface, the w in a small displacement by the reaction

of the surface on the rolling body is zero.

Since the point of contact of the body is for the moment at rest, and so the normal reaction and the force of friction at the point of contact have zero displacements.

(v) The work done by the mutual reaction b/n two bodies of a system in any virtual displacement of the system is zero.

Since the action & reaction are equal and opposite and so the work done by the action balance that done by the reaction.

(vi) If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.

Since, the displacement of the point of application of the force is zero.

* Work done by the tension T of an extensible string of length 'l' during a small displacement

$$= -T.\delta l. \quad [\text{Here, } A'B' = l+\delta l \text{ in case}]$$

* Similarly the work done by the thrust T of an extensible rod of length 'l' during a small displacement

$$= T\,\delta l.$$

* Application of the principle of virtual work

While applying the principle of virtual work we can give any small displacement to the system provided it is consistent with the geometrical condition of the system. This displacement should be such as to exclude the forces which are not required and to include those which are required in the final result.

After giving the displacement we must note the points and the lengths that change and that don't change during the displacement.

If any length or angle is to change during the displacement, we should first find its value in terms of some variable symbol and then after solving the problem we should put its value in the position of =.

In many cases we are required to find the tension of an inextensible string or the thrust/tension of an inextensible rod. In order to find such a tension/thrust we must give the system a displacement in which the length of the string or the rod changes, because otherwise Tension/thrust will not come in the equation of virtual work.

But according to the geometrical condition we cannot give such a displacement to the body. So to get over this difficulty we replace the

string or the rod by two equal and opposite forces T which are equivalent to the tension/thrust in it. By doing so, evidently the equation of virtual work is not affected, while we become free to give the system a displacement in which the length of string/rod changes and consequently T will occur in the equation of virtual work and will thus be determined.

In any problem the virtual work done by the tension T of an extensible string of length l is $-T\delta l$ and the virtual work done by the thrust T of an extensible rod of length l is $+T\delta l$.

In order to find the virtual work done by a force other than a tension or a thrust we first mark a fixed point / fixed straight line. Then we measure the distance of the point of application of the force from this fixed point / line while moving along the line of action of the force. If this distance is x and the force is P, then the virtual work done by the force P during a small displacement is $P\delta x$ in magnitude. If this distance is measured in the direction of the force P, the virtual work done by P is taken with +ve sign, and if the distance x is measured in the direction opposite to that of the force P the virtual work done by P is taken with negative sign.

Equating to zero the total sum of the virtual work done by the forces, we get the equation of the virtual work. Solving this equation we get the value of the required thing to be determined.

If $f(x)$ is a function of x, then during a small displacement in which x changes to $x+\delta x$, we have,

$$\delta f(x) = f(x+\delta x) - f(x)$$

$$= f(x) + \frac{\delta x}{1!} f'(x) + \ldots - f(x)$$

$$= f'(x) \cdot \delta x$$

In many cases the only forces that remain in the equation of virtual work are those due to gravity. In such cases if W is the total weight and $\bar{z}$ the height or depth of its point of application / centre of gravity of the system, above / below a fixed horizontal level then by the Principle of VW for the body, we must have

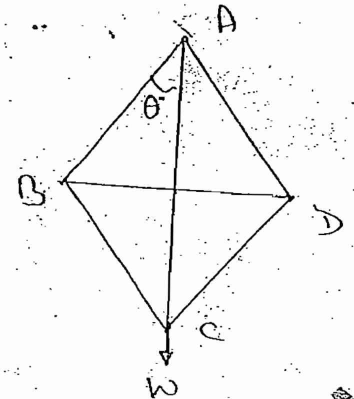
$$W \,\delta\bar{z} = 0$$

i.e. $\quad d\bar{z} = 0$

 (9)

I.A.S 2002 five weightless rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. if a weight 'W' be attached to C and the system be suspended from A, show that there is a thrust in BD equal to $\frac{W}{\sqrt{3}}$

Sol:



The system is suspended from A, and a weight W is attached to C.

$\therefore$ Reaction at A = weight at C

$\therefore$ AC must be vertical.

Let T = thrust in the rod BD

& 2a = length of each rod

i.e. AB = BC = CD = BD = 2a (in $\rightleftharpoons$)

Let $\angle BAC = \theta$

$\therefore$ in $\triangle ABD$, AB = AD = BD

$\therefore \theta = 30°$.

To find the thrust T in BD we shall have to give the system a displacement in which BD must change.

Replace BD by two equal & opposite force T as shown in fig. and then the distance BD can be changed.

Now give small displacement $\delta\theta$ clockwise as shown in fig.

Now,

$BD = 2BO = 2AB\sin\theta = 4a\sin\theta$

& $AC = 2AO = 2AB\cos\theta = 4a\cos\theta$

$\therefore$ By the principal of virtual work, we have,

$T\,\delta(4a\sin\theta) + W\,\delta(4a\cos\theta) = 0$

$\Rightarrow T\cdot 4a\cos\theta\,\delta\theta - W\cdot 4a\sin\theta\cdot\delta\theta = 0$

$\Rightarrow 4a(T\cos\theta - W\sin\theta)\,\delta\theta = 0$

$\delta\theta \neq 0$

$\therefore T\cos\theta = W\sin\theta$

$\Rightarrow T = W\tan\theta$

$\therefore \theta = 30°$

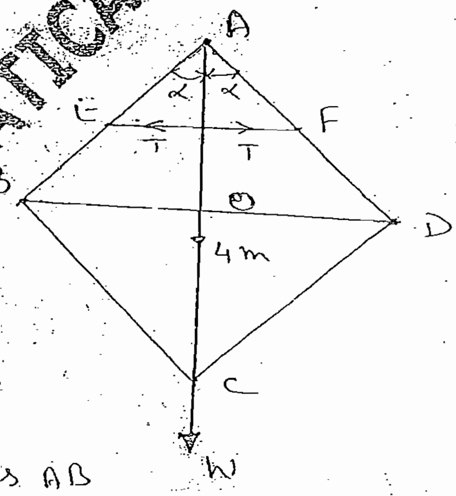
$\therefore T = \frac{W}{\sqrt{3}}$ Proved

(2). Four rods of equal weight m form a rhombus ABCD with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attach to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that thrust in this stiffening rod is

$$(2W + 4m)\tan\alpha$$

Sol^n :

Let ABCD is a framework formed of four equal rods each of weight m length $2a$ (say)



W = weight attached at C

Let EF = light rods joins the rods AB and AD.

Here, AO = AC is $\perp$ to BD as ABCD is rhombus.

& $\angle BAC = \angle DAC = \alpha$.

Let T = Thrust in rod EF

Total weight of All rod = 4m (will act at point O)

Now replacing rod EF by two equal and opposite thrust T as shown in figure.

Give small $\delta\alpha$ displacement about A

$$EF = 2AE \sin\alpha$$
$$= 2a \sin\alpha$$

$$AO = AB\cos\alpha$$
$$= 2a\cos\alpha$$

and $$AC = 2AO$$
$$= 4a\cos\alpha$$

Now, By the principle of virtual work, we have,

$$T \cdot \delta(2a\sin\alpha) + 4m\,\delta(2a\cos\alpha) + W\,\delta(4a\cos\alpha) = 0$$

$$\Rightarrow T \cdot 2a\cos\alpha \cdot \delta\alpha + 4m(-2a\sin\alpha)\cdot\delta\alpha + W(-4a\sin\alpha)\cdot\delta\alpha = 0$$

$$\Rightarrow 2a\{T\cdot\cos\alpha - 4m\sin\alpha - 2W\sin\alpha\}\delta\alpha = 0$$

$$\therefore \delta\alpha \neq 0$$

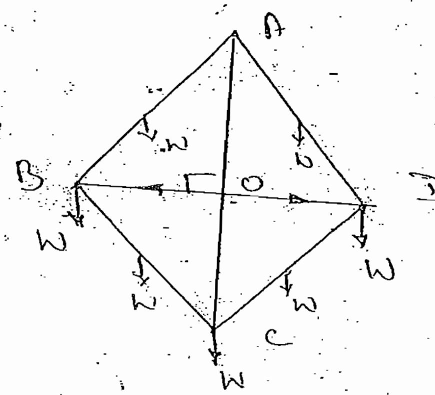
$$\therefore T\cos\alpha - 4m\sin\alpha - 2W\sin\alpha = 0$$

$$\therefore T = (2W + 4m)\tan\alpha$$

Proved.

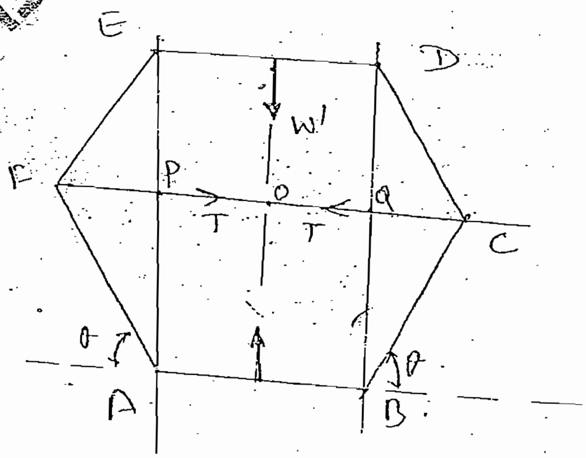
Q. 3

A Square framework, formed of uniform heavy rods of equal weights W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.



Hints : $\theta = 45°$

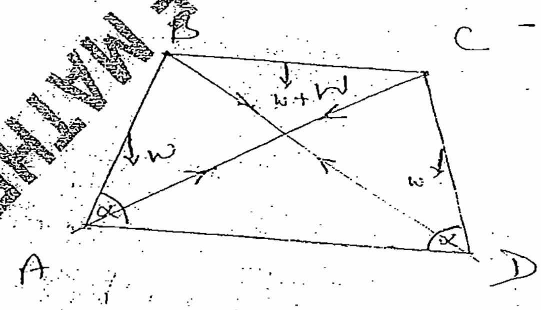
Q. 4 A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely joined together. The two opposite angles C and F are connected by a string which is horizontal, AB being in contact with a horizontal plane. A weight W' is placed at the middle point of DE. Show that the tension of the string is $(3W + W')/\sqrt{3}$



Q.5. Three equal uniform rods AB, BC, CD each of weight w are freely jointed at B and C, and rest in a vertical plane A and D being in a contact with a smooth horizontal table. Two equal light string AC and BD help to support the frame work so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W be placed on BC at its middle point the tension of each string will be of magnitude

$$\left(w + \frac{1}{2}W\right)\cos\alpha \cdot \operatorname{cosec}\frac{\alpha}{2}$$

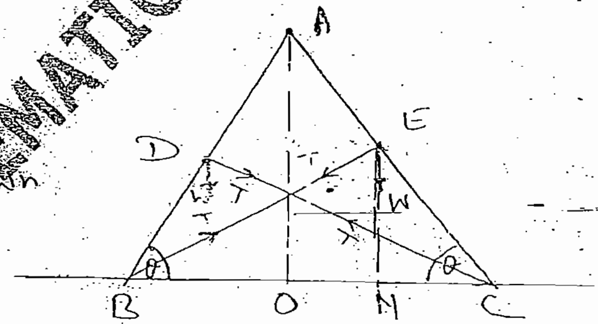
Hint:
fixed level = AD



Q6. Two equal beams AC and AB, each of weight W, are connected by a hinge at A and are placed in vertical plane with their extremities B and C resting on a smooth horizontal plane. They are prevented from falling by strings connecting B and C with the middle points of the opposite beams. Show that the tension of each string is $\frac{1}{8}W\sqrt{1+9\cot^2\theta}$

where 'θ' is the inclination of each beam to the horizon.

Soln: Let AB and AC are equal beams, each of weight W is hinged at 'A' as shown in figure.



Let BE and DC are two strings where D and E are middle points of respective side.

Let length of beam AB or AC = 1

Tension in each string = T

& $\angle ABC = \angle ACB = \theta$

Here the fixed level is horizontal line BC.

$\therefore$ Height of point ~~D~~ or E, above BC

$EM = \frac{1}{2}\sin\theta$

and from $\triangle ABC$

$BM = \frac{3}{4}BC$ $\left[\because \text{in } \triangle ACO,\ E \text{ is mid point of } AC \text{ and } EM \parallel AO,\ M \text{ is mid point of } OC\right.$

$$\Rightarrow BN = \frac{3}{4} \cdot 2l \cdot \cos\theta$$

$$= \frac{3}{2} l \cos\theta$$

Now in $\Delta\, BEN$

$$BE^2 = EN^2 + BN^2$$

$$= \left(\frac{l}{2}\sin\theta\right)^2 + \left(\frac{3}{2} l \cos\theta\right)^2$$

$$\Rightarrow BE^2 = \frac{l^2}{4}(1-\cos^2\theta) + \frac{9}{4} l^2 \cos^2\theta$$

$$\Rightarrow BE^2 = \frac{l^2}{4}(1 + 8\cos^2\theta)$$

$$\therefore BE = \frac{l}{2}\sqrt{1+8\cos^2\theta}$$

Let the system be given small displacement in which θ changes to $\theta+\delta\theta$. The level of the line BC laying on the horizontal plane remains fixed and the points B and C move on the line. The point D and E are slightly displaced.

The Equation of virtual work is

$$-2T\delta(BE) - 2W\,\delta(EN) = 0$$

$$T\,\delta\left(\frac{l}{2}\sqrt{1+8\cos^2\theta}\right) + W\,\delta\left(\frac{l}{2}\sin\theta\right) = 0$$

$$\Rightarrow T \cdot \frac{l}{2}\,\frac{1}{2\sqrt{1+8\cos^2\theta}} \cdot 16\cos\theta\cdot(-\sin\theta)\,\delta\theta + \frac{Wl}{2}\cos\theta\,\delta\theta = 0$$

$$\Rightarrow \left(T \cdot \frac{4\cos\theta(-\sin\theta)}{\sqrt{1+8\cos^2\theta}} + \frac{W}{2}\cos\theta\right)\delta\theta = 0$$

$$\Rightarrow T = \frac{W}{2} \cdot \frac{\sqrt{1+8\cos^2\theta}}{4\sin\theta}$$

$$= \frac{W}{8} \sqrt{\frac{\sin^2\theta + 9\cos^2\theta}{\sin^2\theta}}$$

$$\Rightarrow T = \frac{W}{8}\sqrt{1+9\cot^2\theta}$$

Proved

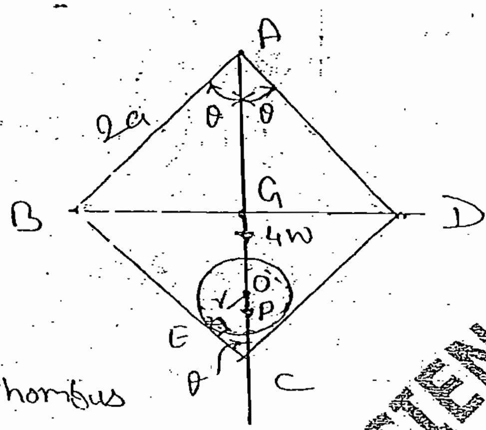
Q.7. Four equal uniform bars each of weight W, are jointed together so as to form a rhombus. This is suspended vertically from one of the joints, and a sphere of weight P, is balanced inside the rhombous so as to keep it from collapsing. Show that if 2θ be the angle at the fixed joint in the figure of equilibrium then

$$\frac{\sin^3\theta}{\cos\theta} = \frac{P r}{4(P+2W)a}$$

where r is the radius of the sphere and $2a$ is the length of each bar

Soln: A rhombous ABCD formed of four rods each of weight W and length $2a$ is suspended from point A.

A sphere of weight P and radius r is placed inside the rhombus, as shown in figure.



The diagonal AC of rhombus must be vertical.

We have, $\angle BAC = \angle DAC = \theta$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta + \delta\theta$.

Now, $AG = 2a\cos\theta$

and ~~depth of O,~~

$AO = AC - OC$

$= 4a\cos\theta - r\,\mathrm{cosec}\,\theta$

$\left[\because \sin\theta = \frac{r}{OC}\right]$

The equation of virtual work

$4W\,\delta(AG) + P\,\delta(AO) = 0$

$\Rightarrow 4W\,\delta(2a\cos\theta) + P\,\delta(4a\cos\theta - r\,\mathrm{cosec}\,\theta) = 0$

$\Rightarrow 4W\cdot 2a(-\sin\theta)\cdot\delta\theta + P(-4a\sin\theta + r\,\mathrm{cosec}\,\theta\cdot\cot\theta)\delta\theta = 0$

$$\Rightarrow \left\{-8Wa\sin\theta + P\left(-4a\sin\theta + r\,\mathrm{cosec}\,\theta.\cot\theta\right)\right\}\delta\theta = 0$$

$$\Rightarrow -8Wa\sin\theta + P\left(-4a\sin\theta + r.\mathrm{cosec}\,\theta.\cot\theta\right) = 0$$

$$\Rightarrow 4a\sin\theta(2W+P) = Pr.\mathrm{cosec}\,\theta.\cot\theta$$

$$\Rightarrow 4a\sin\theta = \frac{Pr}{2W+P}.\frac{\cos\theta}{\sin^2\theta}$$

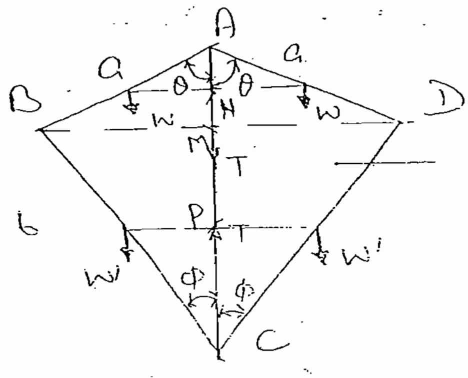
$$\Rightarrow \frac{\sin^3\theta}{\cos\theta} = \frac{Pr}{4a(2W+P)}$$ Proved

(5) A quadrilateral ABCD formed of four uniform rods freely joined to each other at their ends, the rods AB, AD being equal and also the rods BC, CD is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is $(W+W')\sin^2\theta + W'$

where W is the weight of upper rod and W' lower rod and 2θ is equal to the angle BAD.

Soln:

The quadrilateral is suspended from the point A.



Let $AB = AD = a$

and $BC = CD = b$

The diagonal AC must be vertical and BD horizontal.

Let T = tension in the string AC.

Let $\angle BAC = \angle DAC = \theta$

and $\angle BCA = \angle DCA = \phi$

and in $\rightleftharpoons$ position

$$\theta + \phi = 90^\circ \quad \text{— (1)}$$

Now give the system a small symmetrical displacement about AC,

we have,

$$AC = AM + MC = a\cos\theta + b\cos\phi$$

The depth of middle point of AB or AD, below A

$$= AN = \frac{a}{2}\cos\theta$$

and the depth of the middle point of CB or CD below A

$$= AP = a\cos\theta + \frac{b}{2}\cos\phi$$

Now the equation of virtual work

$$-T\,\delta(AC) + 2W\,\delta(AN) + 2W'\,\delta(AP) = 0$$

$$\Rightarrow -T\delta(a\cos\theta + b\cos\phi) + 2W\delta\left(\frac{a}{2}\cos\theta\right) + 2W'\delta\left(a\cos\theta + \frac{b}{2}\cos\phi\right) = 0$$

$$\Rightarrow T(a\sin\theta\,\delta\theta + b\sin\phi\,\delta\phi) - W a\sin\theta\,\delta\theta - 2W'\left(a\sin\theta\,\delta\theta + \frac{b}{2}\sin\phi\,\delta\phi\right) = 0$$

$$\Rightarrow \{T\cdot a\sin\theta - Wa\sin\theta - 2W' a\sin\theta\}\delta\theta = \{-Tb\sin\phi + W' b\sin\phi\}\delta\phi \quad \text{---(1)}$$

From $\triangle ABD$ and $\triangle BCD$

$a\sin\theta = b\sin\phi$

or, $a\cos\theta\,\delta\theta = b\cos\phi\,\delta\phi$ ——— (2)

From (1) and (2), we have,

$$\frac{T\cdot a\sin\theta - W a\sin\theta - 2W'\sin\theta\cdot a}{a\cos\theta} = \frac{-Tb\sin\phi + W' b\sin\phi}{b\cos\phi}$$

$$T\left(\frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi}\right) = (W + 2W')\tan\theta + W'\tan\phi$$

$$\Rightarrow T\cdot\frac{\sin\theta\cdot\cos\phi + \sin\phi\cos\theta}{\cos\theta\cdot\cos\phi} = (W+W')\frac{\sin\theta}{\cos\theta} + W'\left(\frac{\sin\theta}{\cos\theta} + \frac{\sin\phi}{\cos\phi}\right)$$

$$\Rightarrow T\cdot\sin(\theta+\phi) = (W+W')\sin\theta\cdot\cos\phi + W'\sin(\theta+\phi)$$

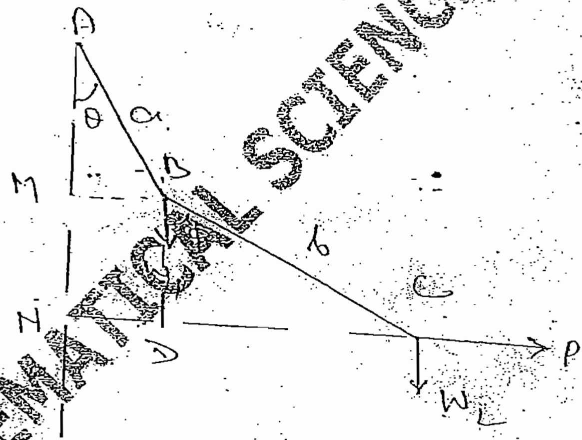
For $\rightleftharpoons$, $\theta + \phi = 90^\circ$

$$\therefore T = (W+W')\sin^2\theta + W' \quad \text{Proved}$$

Q.9.

Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C, the end A is fixed. Prove that, if a horizontal force P is applied at C and in $\rightleftharpoons$ AB, BC are inclined at angle θ, ϕ to the vertical, then

$$P = (W_1 + W_2)\tan\theta$$
$$= W_2 \tan\phi$$



Hint:

$\theta \neq f(\phi)$

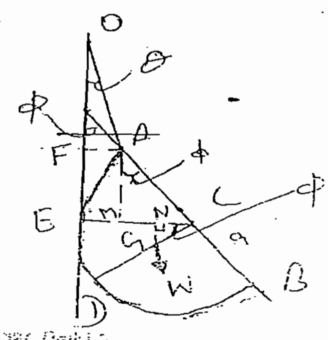
(i) when $\theta \to \theta + \delta\theta$
$\delta\phi = 0$

(ii) when $\phi \to \phi + \delta\phi$
$\delta\theta = 0$

Q.9. IAS-2010, IFoS-2009, 2011

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on the smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\phi = \frac{3}{8} + \tan\theta.$$



Soln:

Let O be the fixed on the wall to which one end of the string is attached.

Let. l = length of the string AO,

a = radius of the hemisphere.

G = point where centre of gravity of the sphere acts, such that

$CG = \frac{3}{8}a$.

θ = angle made by the string on vertical wall.

ϕ = angle made by the base DA of hemisphere on vertical wall.

Now,

$OG = OF + FM + NG$

$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$. the point remains fixed and the length of the string AO doesnot change.

$\therefore$ the equation of virtual equation is.

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0$$

or, $-l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$

or, $l\sin\theta\,\delta\theta = \left(\frac{3}{8}a\cos\phi - a\sin\phi\right)\delta\phi$ —(i)

From the figure,

$E- = a$

$\Rightarrow EH + MC = a$

$\Rightarrow FA + MC = a$

$\Rightarrow l\sin\theta + a\sin\phi = a$

$\Rightarrow l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi = 0$

or, $-l\cos\theta\,\delta\theta = +a\cos\phi\,\delta\phi$ —②

Dividing ① by ②, we have,

$-\tan\theta = \frac{3}{8} \ldots \sin\phi$

$\Rightarrow -\tan\phi = \ldots + \tan\theta$

— Proved

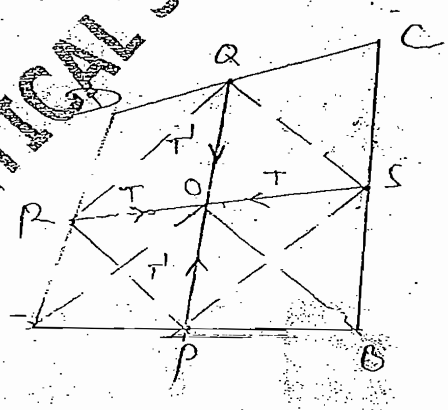
10). The middle point of the opposite sides of a jointed quadrilateral are connected by light rods of length l, l'. If T, T' be the tensions in these rods, prove that

AS 2001, 2006

$$\frac{T}{l} + \frac{T'}{l'} = 0$$

Soln:

Let ABCD be a quadrilateral, whose middle points of opposite sides are joined by two strings of length l and l'.



Since, P, Q, R, S is the middle point

PSQR is a parallelogram

Now, Replace the string PQ with two equal and opposite forces T

and replace string RS with two equal and opposite forces $-T'$

Now give small displacement in which PQ and RS changes slightly.

The length of the rods AB, BC, CD, DA do not change.

The Equation of virtual work

$$-T\,\delta(RS) - T'\,\delta(PQ) = 0 \quad -(1)$$

$\Rightarrow$ in Δ OAB

OP is median

$$\therefore OA^2 + OB^2 = 2OP^2 + 2AP^2$$

$$= 2\left(\tfrac{1}{2}PQ\right)^2 + 2\left(\tfrac{1}{2}AB\right)^2$$

$$= \frac{PQ^2 + AB^2}{2} \quad -(2)$$

Similarly in Δ OCD

$$OC^2 + OD^2 = \tfrac{1}{2}(PQ^2 + CD^2) \quad -(3)$$

From (2) + (3),

$$OA^2 + OB^2 + OC^2 + OD^2 = \tfrac{1}{2}(2PQ^2 + AB^2 + CD^2) \quad -(4)$$

Similarly for Δ AOD and BOC, we have,

$$OA^2 + OB^2 + OC^2 + OD^2 = \tfrac{1}{2}(2RS^2 + AD^2 + CD^2) \quad -(5)$$

From (4) and (5),

$$2PQ^2 + AB^2 + CD^2 = 2RS^2 + AD^2 + CD^2$$

or, $2(PQ^2 - RS^2) = \text{constant} \quad -(6)$

[Since, AB, BC, CD, AD are all of fixed length]

or, $PQ\,\delta(PQ) = RS\,\delta(RS)$

$$\Rightarrow \frac{S(RS)}{S(PQ)} = \frac{PQ}{RS}$$

But, $RS = l$, $PQ = l'$ given. at $\Rightarrow$

$$\therefore \frac{S(RS)}{S(PQ)} = \frac{l'}{l}$$

$\therefore$ from (i),

$$\frac{T}{T'} = -\frac{S(PQ)}{S(RS)} = -\frac{l}{l'}$$

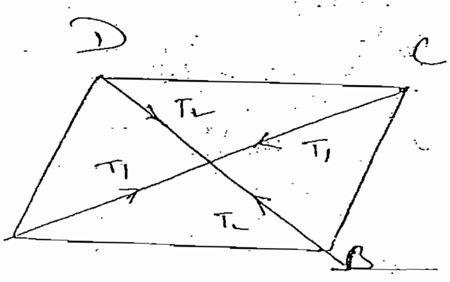
$$\Rightarrow \frac{T}{l} + \frac{T'}{l'}$$

Q. 11) Four rods are jointed together to form a parallelogram, the opposite joints are joined by the strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.

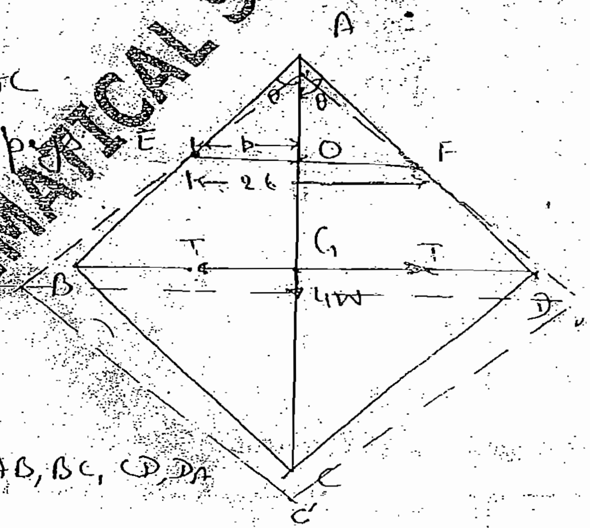
Sol^n: $\rightarrow$ Here gives only linear displacement

Hint: in parallelogram,

$$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2 = \text{Constant}$$



Q12. Four equal rods, each of length 2a and weight W, are freely jointed to form a square ABCD, which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance 2b apart on the same level. Find the stress in the rod BD.



Soln: Let the rods AB and AC rest on two fixed smooth pegs E and F, which are at the same level and

$EF = 2b$

Let $2a$ = length of each rods, AB, BC, CD, DA

T = thrust in the rod BD

$\angle BAC = \angle DAC$

Now Replace the rod BD by two equal and opposite forces T as shown in figure.

Now System is given small displacement in which θ changes to $\theta + \delta\theta$

Here our refrence line is EF.

forces contributing to the virtual works are

(i) The thrust in the rod BD

(ii) the weight 4W acting at G.

$BD = 2 \cdot 2a \sin\theta$

$= 4a \sin\theta$

and

$OG = AG - AO$ $\left[\begin{array}{l}\text{in } \triangle AOE \\ \tan\theta = \frac{b}{AO}\end{array}\right]$

$= 2a\cos\theta - b\cot\theta$

$\therefore$ The Equation of virtual work is

$T\,\delta(BD) + 4W\,\delta(OG) = 0$

$\Rightarrow T\,\delta(4a\sin\theta) + 4W\,\delta(2a\cos\theta - b\cot\theta) = 0$

$\Rightarrow T\,4a\cos\theta\,\delta\theta + 4W(-2a\sin\theta + b\,\mathrm{cosec}^2\theta)\,\delta\theta = 0$

$\left\{T\cdot 4a\cdot\cos\theta + 4W(b\,\mathrm{cosec}^2\theta - 2a\sin\theta)\right\}\delta\theta = 0$

$\because \delta\theta \neq 0$

$T \cdot 4a\cos\theta + 4W(b\,\mathrm{cosec}^2\theta - 2a\sin\theta) = 0$

$\Rightarrow T = \frac{W}{a}\left(2a\tan\theta - \frac{b\,\mathrm{cosec}^2\theta}{\cos\theta}\right)$

But in equilibrium position,

we have $\theta = 45°$

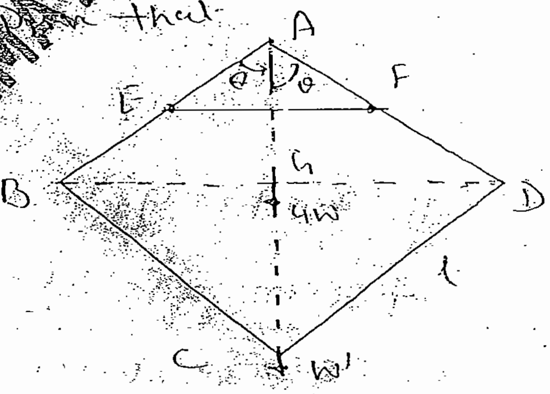
$$\therefore T = \frac{W}{a}(2a - 2b\sqrt{2})$$

$$= \frac{2W}{a}(a - b\sqrt{2})$$

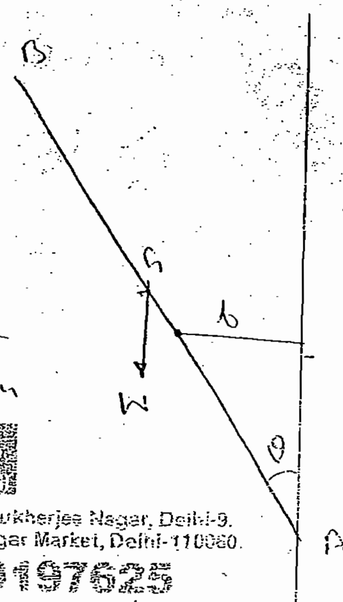
Ans

Q 13. A rhombus is formed of rods each of weight W and length 'l' with smooth joints. It rests symmetrically with its two upper sides in contact with two smooth pegs at the same level and at a distance 2a apart. A weight W' is hung at the lowest point; if the sides of the rhombus make an angle θ with the vertical, then that

$$\sin^3\theta = \frac{a(4W + W')}{l(4W + 2W')}$$

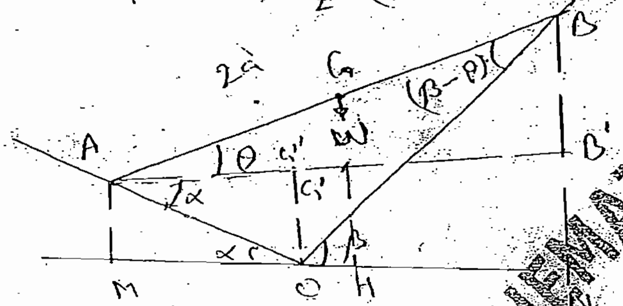


Q. 14. A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance 'b' from the wall. Show that in position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$



Q. 15. A heavy uniform rod, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that

$$\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$$



Hints $AB' = 2a\cos\theta$, $BB' = 2a\sin\theta$

$\therefore GG' = a\sin\theta$. —①

$$\frac{OA}{\sin(\beta-\theta)} = \frac{2a}{\sin(\pi-(\alpha+\beta))} \Rightarrow OA = 2a\cdot\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}$$

$$\therefore AM = G'O = OA\sin\alpha = 2a\cdot\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}\cdot\sin\alpha.$$

$$\therefore GH = GG' + G'H$$

$$= a\sin\theta + 2a\cdot\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)}\cdot\sin\alpha.$$

$\therefore \delta(GH) = 0$

$$\Rightarrow -a\cdot\cos\theta + 2a\cdot\frac{-\cos(\beta-\theta)}{\sin(\alpha+\beta)}\cdot\sin\alpha = 0$$

$$\Rightarrow \cos\theta\cdot\sin(\alpha+\beta) = 2\cdot\sin\alpha\cdot\cos(\beta-\theta)$$

$$\Rightarrow \sin(\alpha+\beta) = 2\cdot\sin\alpha\,(\cos\beta + \tan\theta\cdot\sin\beta)$$

$$\Rightarrow \tan\theta = \left\{\frac{\sin(\alpha+\beta)}{2\sin\alpha} - \cos\beta\right\}\frac{1}{\sin\beta}$$

$$\Rightarrow \tan\theta = \left\{ \frac{\sin(\alpha+\beta)}{2\cdot\sin\alpha\cdot\sin\beta} - \cot\beta \right\}$$

$$= \left\{ \frac{\sin\alpha\cdot\cos\beta + \sin\beta\cdot\cos\alpha}{2\sin\alpha\cdot\sin\beta} - \cot\beta \right\}$$

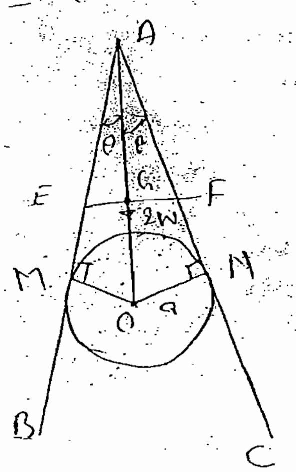
$$= \frac{1}{2}\left\{ \cot\beta + \cot\alpha - 2\cot\beta \right\}$$

$$= \frac{1}{2}\left\{ \cot\alpha - \cot\beta \right\}$$

Q. 16 Two equal rods, AB and AC each of length 2b are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then

$$b\sin^3\theta = a\cos\theta$$

Sol: Let O be the centre of the given fixed circle and W be the ~~weight of each of the rods AB and~~ AC.



Then AO is vertical.

Therefore, $\angle BAO = \theta = \angle CAO$

Give the rods a small displacement in which θ changes to $\theta + \delta\theta$. The point O remains fixed

From figure,

$$GO = OA - AG \qquad \left[\sin\theta = \frac{OM}{OA} \Rightarrow OA = a\,\text{cosec}\,\theta \right.$$
$$\left. AG = b\cos\theta \right.$$

$GO = a\,\text{cosec}\,\theta - b\cos\theta$

∴ Equation of virtual work

$-2W\,\delta(OG) = 0$

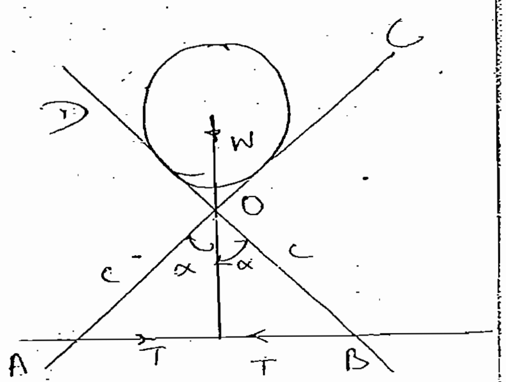
$\Rightarrow \delta(a\,\text{cosec}\,\theta - b\cos\theta) = 0$

$\Rightarrow -a\,\text{cosec}\,\theta\cdot\cot\theta + b\sin\theta = 0$

$\Rightarrow b\sin^3\theta = a\cos\theta$ Proved

Q.11 Two light rods AOC, BOD are smoothly hinged at O, a point at a distance 'c' from each of the ends A, B which are connected by a string of length $2c\sin\alpha$. The rods rest in a vertical plane with the ends A and B on a smooth horizontal table. A smooth circular disc of radius 'a' and weight W is placed on the rods above 'O' with its plane vertical & that rods are tangents to the disc. Prove that the tension of the string is

$\frac{1}{2}W\left\{\frac{a}{c}\,\text{cosec}^2\alpha + \tan\alpha\right\}$



Q. 18 One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are jointed by an elastic string, of natural length 'a' and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2\sin^{-1}(3/4)$.

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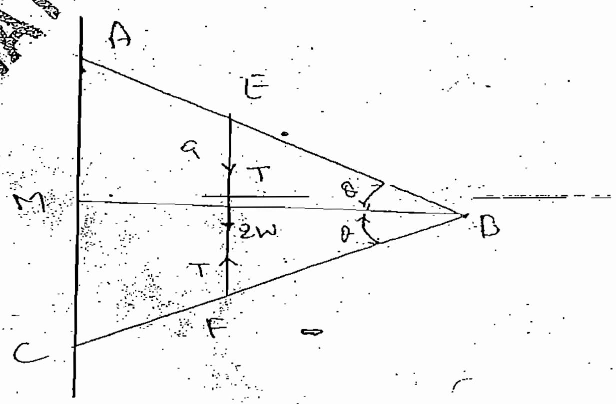
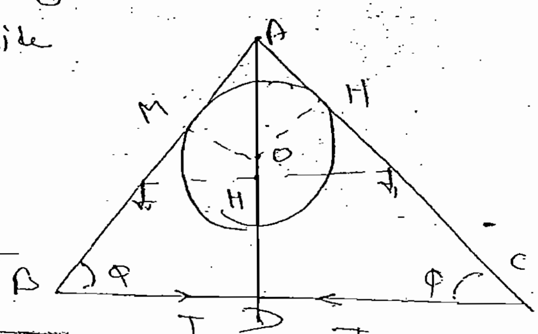


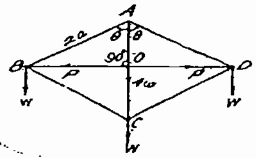
Q 19. Two equal rods, each of weight W and length 'l', are hinged together and placed astride a smooth horizontal cylindrical peg of radius 'r'. Then the lower ends are tied together by a string and the rods are left at the same inclination φ to the horizontal. Find the tension in the string. If the string is just slack, show that [illegible] $\frac{l}{2r}$



Ex.20 *Four equal uniform rods, each of weight w, are freely jointed to form a rhombus $ABCD$. The frame work is suspended freely from A and a weight W is attached to each of the joints B, C, D. If two horizontal forces each of magnitude P acting at B and D keep the angle BAD equal to 120°, prove that*

$$P=(W+w)\,2\sqrt{3}.$$

Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended from the point A and a weight W is attached to each of the points B, C and D. To save the system from collapsing two horizontal forces each of magnitude P act at B and D and in equilibrium $\angle BAD=120°$. Obviously the nature of the forces P is like that of thrust.



The total weight $4w$ of all the four rods AB, BC, CD and DA can be taken acting at the point of intersection O of the diagonals AC and BD. Obviously the line AC must be vertical and so BD is horizontal.

To find P we have to give the system a displacement in which the length BD must change and consequently the angle BAD will change so let us assume that $\angle BAC=\theta=\angle DAC$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and we shall measure the distances of the points of application of various forces from the point A. The points B, C, D and O change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes. The angle AOB will remain 90°.

We have

$$BD=2BO=2AB\sin\theta=4a\sin\theta,$$

the depth of B or D or O below A

$$=AO=2a\cos\theta,$$

and the depth of C below A

$$=AC=2AO=4a\cos\theta.$$

By the principle of virtual work, we have

$$P\delta\,(4a\sin\theta)+4w\delta\,(2a\cos\theta)+2W\delta\,(2a\cos\theta)$$
$$+W\delta\,(4a\cos\theta)=0$$

or $\quad 4a\,P\cos\theta\,\delta\theta-8aw\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$

or $\quad 4a\,\{P\cos\theta-2w\sin\theta-W\sin\theta-W\sin\theta\}\,\delta\theta=0$

or $\quad P\cos\theta-2\,(W+w)\sin\theta=0$

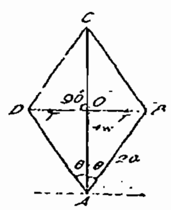
or $\quad P=2\,(W+w)\tan\theta.$

But in the position of equilibrium, $\theta=60°$.

$\therefore\quad P=2\,(W+w)\tan 60°=2\,(W+w)\sqrt{3}=(W+w)\,2\sqrt{3}.$

Ex.21 *Four equal uniform rods, each of weight W, are jointed to form a rhombus $ABCD$, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC=\theta$ by a light string joining B and D. Find the tension of the string.*

Sol. $ABCD$ is a framework formed of four equal rods each of weight W and say of length $2a$. It is placed in a vertical plane with AC vertical and A resting on a horizontal plane. To keep the system in the form of a rhombus a light string joins B and D and prevents the points B and D from moving in the directions OB and OD respectively. Let T be the tension in the string BD. The total weight $4W$ of all the four rods may be taken acting at the point of intersection O of the diagonals AC and BD.



Let $\quad \angle DAC=\theta=\angle BAC.$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A resting on the horizontal plane remains fixed. The points B, C, D and O will change. The lengths of the rods AB, BC, CD and DA will remain fixed while the length BD will change. The angle DOC will remain 90°.

We have $\quad BD=2BO=2AB\sin\theta=4a\sin\theta,$

and the height of O above the fixed point A

$$=AO=2a\cos\theta.$$

By the principle of virtual work, we have

$$-T\delta\,(4a\sin\theta)-4W\delta\,(2a\cos\theta)=0. \qquad ...(1)$$

[Note that in the equation (1) the work done by the weight $4W$ has ~~been taken~~ with negative sign because the distance AO of its point of application O from the fixed point A is in a direction opposite to the direction of $4W$.]

From the equation (1), we have

$$-4a\,T\cos\theta\,\delta\theta+8aW\sin\theta\,\delta\theta=0$$

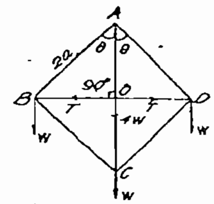
or $\quad 4a\,[-T\cos\theta+2W\sin\theta]\,\delta\theta=0$

or $\quad -T\cos\theta+2W\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=2W\tan\theta.$

Ex.22 *A square framework, formed of uniform heavy rods of equal weight W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.*

Sol. $ABCD$ is a square framework formed of four rods each of weight W and say of length $2a$. It is suspended from the point A and a weight W is suspended from each of the three lower corners B, C and D. A light rod along the horizontal diagonal BD prevents the system from collapsing. Let T be the thrust in the rod BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at O.



To find T we shall have to give the system a displacement in which BD must change. So replace the rod BD by two equal and opposite forces T as shown in the figure and assume that $\angle BAC=\theta=\angle CAD$. [Note that the angle BAC will change during a displacement in which BD is to change.]

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and the points B, O, D and C change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes.

We have $\quad BD=2BO=2AB\sin\theta=4a\sin\theta,$

the depth of each of the points B, C and D below the fixed point A

$$=AO=2a\cos\theta,$$

and the depth of C below $A=2AO=4a\cos\theta.$

By the principle of virtual work, we have

$$T\delta\,(4a\sin\theta)+4W\delta\,(2a\cos\theta)+2W\delta\,(2a\cos\theta)$$
$$+W\delta\,(4a\cos\theta)=0$$

or $\quad 4a\,T\cos\theta\,\delta\theta-8a\,W\sin\theta\,\delta\theta-4a\,W\sin\theta\,\delta\theta-4a\,W\sin\theta\,\delta\theta=0$

or $\quad 4a\,[T\cos\theta-4W\sin\theta]\,\delta\theta=0$

or $\quad T\cos\theta-4W\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$

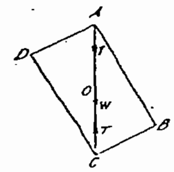
or $\quad T=4W\tan\theta.$

But in the position of equilibrium $\theta=45°$.

$\therefore\quad T=4W\tan 45°=4W=$ the total weight of the four rods.

Ex.23 *Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.*

Sol. $ABCD$ is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC. Let T be the tension in the string AC. The total weight W of all the four rods AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC=2x$.



Give the system a small displacement in which x changes to $x+\delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes. We have, $AO=x$

By the principle of virtual work, we have

$$-T\delta\,(AC)+W\delta\,(AO)=0$$

or $\quad -T\delta\,(2x)+W\delta\,(x)=0$

or $\quad -2T\delta x+W\delta x=0$

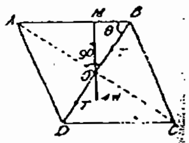
or $\quad [-2T+W]\,\delta x=0$

or $\quad -2T+W=0 \qquad [\because\ \delta x\neq 0]$

or $\quad T=\frac{1}{2}W=\frac{1}{2}$ (total weight of all the four rods).

Ex.24 *A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is*

$$\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$



Sol. $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have $\angle AOB=90°$.

Let $\angle ABO=\theta$. Draw OM perpendicular to AB.

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The $\angle AOB$ will remain $90°$.

We have $BD=2BO=2AB\cos\theta=2b\cos\theta$.

[Note that in the position of equilibrium $BD=a$. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line $AB=MO$.

$=BO\sin\theta=(AB\cos\theta)\sin\theta=b\sin\theta\cos\theta$.

By the principle of virtual work, we have

$$-T\delta(2b\cos\theta)+4W\delta(b\sin\theta\cos\theta)=0$$

or $\quad 2bT\sin\theta\,\delta\theta+4bW(\cos^2\theta-\sin^2\theta)\,\delta\theta=0$

or $\quad 2b\,[T\sin\theta-2W(\sin^2\theta-\cos^2\theta)]\,\delta\theta=0$

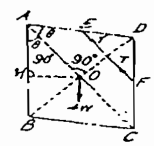
or $\quad T\sin\theta-2W(\sin^2\theta-\cos^2\theta)=0 \qquad [\because \delta\theta\neq 0]$

or $\quad T=\dfrac{2W(\sin^2\theta-\cos^2\theta)}{\sin\theta}=\dfrac{2W(1-2\cos^2\theta)}{\sqrt{(1-\cos^2\theta)}}.$

In the position of equilibrium, $BD=a$ or $BO=\frac{1}{2}a$. So in the position of equilibrium, $\cos\theta=\dfrac{BO}{AB}=\dfrac{\frac{1}{2}a}{b}=\dfrac{a}{2b}$.

$$\therefore\quad T=\frac{2W\{1-2(a^2/4b^2)\}}{\sqrt{\{1-(a^2/4b^2)\}}}=\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

Ex.25 *Four equal uniform rods, each of weight W, are smoothly jointed so as to form a square $ABCD$; the side AB is fixed (clamped) in a vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC. Find the tension of the string.*



Sol. $ABCD$ is a framework formed of four equal uniform rods each of weight W and say of length $2a$. The side AB is fixed in a vertical position with A uppermost. A string joins the middle points E and F of AD and DC respectively and in equilibrium $ABCD$ is a square.

Let T be the tension in the string EF. The total weight $4W$ of all the rods AB, BC, CD and DA acts at O, the point of intersection of the diagonals AC and BD. We have, $\angle AOD=90°$. Let $\angle BAC=\theta=\angle DAC$. Draw OM perpendicular to AB.

[Note that we have drawn $ABCD$ as a rhombus and not as a square because in a displacement in which EF is to change the figure will not remain a square. After finding the value of the tension T we shall use the fact that in the position of equilibrium the figure is a square].

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AB will remain fixed and so A is a fixed point. The points C, D and O will change. The lengths of the rods AB, BC, CD and DA do not change while the length EF changes. The $\angle AOD$ remains $90°$.

We have $EF=\frac{1}{2}AC=AO=AD\cos\theta=2a\cos\theta$. Also the depth of O below the fixed point A i.e., the distance of O from the fixed point A in the direction of the force $4W$

$=AM=AO\cos\theta=(2a\cos\theta)\cos\theta=2a\cos^2\theta$.

By the principle of virtual work, we have

$$-T\delta(2a\cos\theta)+4W\delta(2a\cos^2\theta)=0$$

or $\quad 2aT\sin\theta\,\delta\theta-16aW\cos\theta\sin\theta\,\delta\theta=0$

or $\quad 2a\sin\theta\,(T-8W\cos\theta)\,\delta\theta=0$

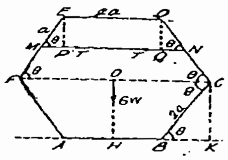
or $\quad T-8W\cos\theta=0 \qquad [\because \delta\theta\neq 0 \text{ and } \sin\theta\neq 0]$

or $\quad T=8W\cos\theta.$

But in the position of equilibrium, $\theta=45°$.

$$\therefore\quad T=8W\cos 45°=8W(1/\sqrt{2})=4W\sqrt{2}.$$

Ex.26 *Six equal heavy beams are freely jointed at their ends to form a hexagon, and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Find its tension in terms of W and θ, where W is the weight of each beam.*



Sol. $ABCDEF$ is a hexagon formed of six equal heavy beams each of weight W and say of length $2a$. The frame is placed in a vertical plane with the beam AB resting on a horizontal plane. To save the system from collapsing the middle points M and N of the beams FE and CD are connected by a light cord. Let T be the tension in the cord MN.

The line FC is horizontal. We have $\angle EFC=\theta=\angle DCF$.

Draw EP and DQ perpendiculars to MN.

The total weight $6W$ of all the six rods can be taken acting at O, the middle point of FC. Draw OH and CK perpendicular to AB. We have $\angle CBK=\theta$.

Give the system a small symmetrical displacement about the vertical line OH in which θ changes to $\theta+\delta\theta$. The line AB on the horizontal plane remains fixed, and the distance of the point of application O of the weight $6W$ will be measured from AB. The lengths of the rods AB, BC etc. remain fixed while the length MN changes. The point O also changes.

We have $\quad MN=MP+PQ+QN$

$$=a\cos\theta+2a+a\cos\theta=2a+2a\cos\theta$$

[Note that $PQ=ED=2a$, because ED remains fixed].

Also the height of O above the fixed line AB

$$=HO=KC=2a\sin\theta$$

By the principle of virtual work, we have

$$-T\delta(2a+2a\cos\theta)-6W\delta(2a\sin\theta)=0$$

[The work done by $6W$ is taken with $-$ive sign because the direction of HO is opposite to that of $6W$]

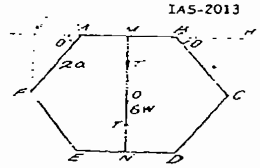
or $\quad 2aT\sin\theta\,\delta\theta-12aW\cos\theta\,\delta\theta=0$

$$\Rightarrow 2a(T\sin\theta-6W\cos\theta)\delta\theta=0$$

$$\Rightarrow T\sin\theta-6W\cos\theta=0 \quad (\because \delta\theta\neq 0)$$

$$\Rightarrow T=6W\cot\theta$$

Ex.27 *Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is $3W$.* IAS-2013



Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let

$$\angle FAK=\theta=\angle CBH.$$

Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length MN changes and the point O also changes.

We have

$$MN=2MO=2KF=2AF\sin\theta=4a\sin\theta.$$

Also the depth of O below the fixed line AB

$$=MO=2a\sin\theta.$$

By the principle of virtual work, we have

$$-T\delta(4a\sin\theta)+6W\delta(2a\sin\theta)=0$$

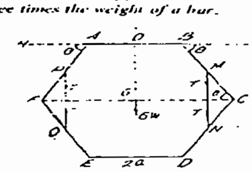
or $\quad -4aT\cos\theta\,\delta\theta+12aW\cos\theta\,\delta\theta=0$

or $\quad 4a\,[-T+3W]\cos\theta\,\delta\theta=0$

or $\quad -T+3W=0 \qquad [\because \delta\theta\neq 0 \text{ and } \cos\theta\neq 0]$

or $\quad T=3W.$

Ex.28 *Six equal bars are freely jointed at their extremities forming a regular hexagon $ABCDEF$ which is kept in shape by vertical strings joining the middle points of BC, CD and AF, FE respectively, the side AB being held horizontal and uppermost. Prove that the tension of each string is three times the weight of a bar.*



Sol. $ABCDEF$ is a hexagon formed of six equal bars say each of weight W and length $2a$. The rod AB is held horizontal and uppermost. The middle points M and N of BC and CD are joined by a string and the middle points P and Q of AF and FE are also joined by a string. Let T be the tension in each of the strings PQ and MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at G, the middle point of FC.

Let $\quad \angle HAF=\theta=\angle ABC.$

Give the system a small symmetrical displacement about the vertical line OG in which θ changes to $\theta+\delta\theta$. The line AB remain fixed. The lengths of the rods AB, BC etc. remain fixed, the lengths MN and PQ change and the point G also changes.

We have

$$PQ=MN=2MC\sin\theta=2a\sin\theta.$$

Also the depth of G below AB

$$=OG=BC\sin\theta=2a\sin\theta.$$

By the principle of virtual work, we have

$-2T\delta(2a\sin\theta)+6W\delta(2a\sin\theta)=0$

or $-4aT\cos\theta\,\delta\theta+12aW\cos\theta\,\delta\theta=0$

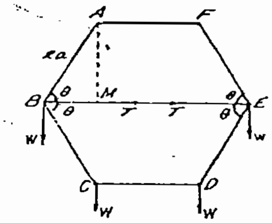
or $4a\cos\theta(-T+3W)\,\delta\theta=0$

or $-T+3W=0$ $[\because \delta\theta\neq0$ and $\cos\theta\neq0]$

or $T=3W$ *i.e.*, the tension of each string is three times the weight of a bar.

Ex.29 *Six equal light rods are joined to form a hexagon ABCDEF which is suspended at A and F so that AF is horizontal. A rod BE, also light, keeps the figure from collapsing and is of such a length that the rods ending in the points B, E are inclined at an angle of 45° to the vertical. Equal weights are suspended from B, C, D, E. Find the stress in BE.*

Sol. $ABCDEF$ is a hexagon formed of six equal light rods say each of length $2a$. It is suspended at A and F so that AF is horizontal. Equal weights W are suspended from each of the points B, C, D and E. A light rod joining B and E saves the system from collapsing. Let T be the stress in the rod BE. Since the rod BE prevents the points B and E from moving inwards, therefore the stress in the rod BE is a thrust.



Let $\angle ABE=\theta=\angle FEB=\angle CBE=\angle DEB$.

Replace the rod BE by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AF remains fixed. The points B, C, D and E change. The lengths of the rods AB, BC etc. do not change while the length BE changes.

We have

$BE=AF+2BM=2a+2.2a\cos\theta=2a+4a\cos\theta$,

the depth of each of the points B and E below AF

$=AM=2a\sin\theta$,

and the depth of each of the points C and D below AF

$=2AM=4a\sin\theta$.

By the principle of virtual work, we have

$T\delta(2a+4a\cos\theta)+2W\delta(2a\sin\theta)+2W\delta(4a\sin\theta)=0$

or $-4aT\sin\theta\,\delta\theta+4aW\cos\theta\,\delta\theta+8aW\cos\theta\,\delta\theta=0$

or $4a(-T\sin\theta+W\cos\theta+2W\cos\theta)\,\delta\theta=0$

or $-T\sin\theta+3W\cos\theta=0$ $[\because \delta\theta\neq0]$

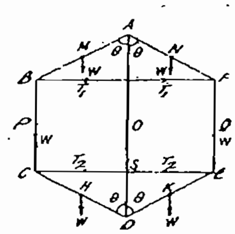
or $T=3W\cot\theta$.

But in the position of equilibrium each of the rods AB, BC, EF and ED makes an angle 45° with the vertical and so also with the horizontal BE. Therefore in the position of equilibrium, $\theta=45°$ and $T=3W\cot45°=3W$.

Ex.30 *Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are $(5\sqrt{3}/2)$ W and $(\sqrt{3}/2)$ W, where W is the weight of each rod.*

Sol. Let the length of each of the rods AB, BC etc. be $2a$ and let θ be the angle which each of the slant rods AB, AF, DC and DE makes with the vertical AD.

Let T_1 and T_2 be the thrusts in the rods BF and CE respectively. Here A is the fixed point. The weights of the rods AB, BC etc. act at their respective middle points as shown in the figure.



Let us first find the thrust T_1.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and keep the rod CF intact so that during any displacement the length CE does not change. Now give the system a small symmetrical displacement about the vertical line AD in which θ at the end A changes to $\theta+\delta\theta$ while θ at the end D does not change. The portion $BCDEF$ moves as it is. The length BF changes while the length CE does not change so that during this small displacement the work done by the thrust T_2 of the rod CE is zero. The centres of gravity of all the six rods AB, BC etc. are slightly displaced.

We have $BF=4a\sin\theta$.

In this case we cannot take the total weight of the rods AB, BC etc. act at the middle point O of AD. The depth of each of the points M and N below A is $a\cos\theta$, the depth of each of the points P and Q below A is $2a\cos\theta+a$, and the depth of each of the points H and K below A is $2a\cos\theta+2a+\frac{1}{2}SD$ where in this case SD is fixed.

By the principle of virtual work, we have

$T_1\delta(4a\sin\theta)+2W\delta(a\cos\theta)+2W\delta(2a\cos\theta+a)$
$+2W\delta(2a\cos\theta+2a+\frac{1}{2}SD)=0$

or $4a\,T_1\cos\theta\,\delta\theta-10aW\sin\theta\,\delta\theta=0$

or $2a(2T_1\cos\theta-5W\sin\theta)\,\delta\theta=0$

or $2T_1\cos\theta-5W\sin\theta=0$ $[\because \delta\theta\neq0]$

or $T_1=\frac{5}{2}W\tan\theta$.

But in the position of equilibrium, the hexagon is a regular one and so $\theta=\pi/3$.

Therefore $T_1=\frac{5}{2}W\tan\frac{1}{3}\pi=\frac{5}{2}W\sqrt{3}$.

Now let us proceed to find the thrust T_2.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and so replace the rod CE by two equal and opposite forces T_2 as shown in the figure. Give the system a small symmetrical displacement about the vertical line AD in which θ at both the ends A and D changes to $\theta+\delta\theta$ so that both the lengths BF and CE change. In this case the total weight $6W$ of all the six rods AB, BC etc. can be taken acting at the middle point O of AD.

We have

$BF=4a\sin\theta$, $CE=4a\sin\theta$ and $AO=2a\cos\theta+a$.

By the principle of virtual work, we have

$T_1\delta(4a\sin\theta)+T_2\delta(4a\sin\theta)+6W\delta(2a\cos\theta+a)=0$

or $4a\,T_1\cos\theta\,\delta\theta+4a\,T_2\cos\theta\,\delta\theta-12aW\sin\theta\,\delta\theta=0$

or $4a[(T_1+T_2)\cos\theta-3W\sin\theta]\,\delta\theta=0$

or $(T_1+T_2)\cos\theta-3W\sin\theta=0$ $[\because \delta\theta\neq0]$

or $T_1+T_2=3W\tan\theta$.

But in the position of equilibrium $\theta=\pi/3$.

$\therefore$ $T_1+T_2=3W\tan\frac{1}{3}\pi=3W\sqrt{3}$.

$\therefore$ $T_2=3W\sqrt{3}-T_1=3W\sqrt{3}-\frac{5W\sqrt{3}}{2}=\frac{W\sqrt{3}}{2}$.

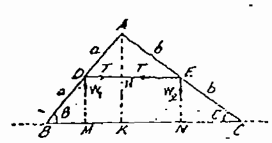
Ex.31 *Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is*

$$\frac{W}{\tan B+\tan C}$$

where W is the total weight of the rods AB and AC.

[Gorakhpur 79; Jiwaji 78]

Sol. AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let



$AB=2a$ and $AC=2b$.

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore the total weight $W=W_1+W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B+\delta B$ and C changes to $C+\delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move.

We have

$DE=DH+HE=a\cos B+b\cos C$,

the height of any point of the line DE above BC

$=DM=a\sin B$.

The equation of virtual work is

$-T\delta(a\cos B+b\cos C)-W\delta(a\sin B)=0$

or $aT\sin B\,\delta B+bT\sin C\,\delta C-aW\cos B\,\delta B=0$

or $a(W\cos B-T\sin B)\,\delta B=bT\sin C\,\delta C$. ...(1)

From the figure,

$DM=a\sin B$ and $EN=b\sin C$.

Since $DM=EN$, therefore $a\sin B=b\sin C$.

$\therefore$ $\delta(a\sin B)=\delta(b\sin C)$

or $a\cos B\,\delta B=b\cos C\,\delta C$. ...(2)

Dividing (1) by (2), we have

$$\frac{W\cos B - T\sin B}{\cos B} = \frac{T\sin C}{\cos C}$$

or $\quad W - T\tan B = T\tan C$

or $\quad T(\tan B + \tan C) = W$

or $\quad T = \dfrac{W}{\tan B + \tan C}$

Ex.32. *Two uniform rods AB, BC of weights W' and W' are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is*

$$(W + W')\cos A\cos C/\sin B.$$

Find the additional tension in the cord caused by suspending a weight W'' from B.

Sol. Draw figure and proceed as in Ex. 23.

In the first case, we shall get

$$T = \frac{(W+W')}{\tan A + \tan C} = \frac{(W+W')\cos A\cos C}{\sin A\cos C + \cos A\sin C}$$

$$= \frac{(W+W')\cos A\cos C}{\sin(A+C)} = \frac{(W+W')\cos A\cos C}{\sin(180° - B)}$$

$$= \frac{(W+W')\cos A\cos C}{\sin B}$$

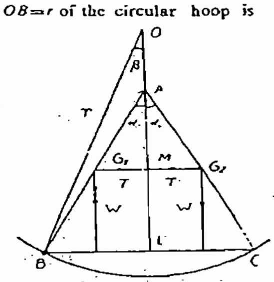
In the second case when a weight W'' is also suspended from B, let T' be the tension in the cord. Write the new equation of virtual work and find T'.

The required additional tension in the cord

$$= T' - T = (2W''\cos A\cos C)/\sin B.$$

Ex.33. *Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that if the string is stretched, its tension is W (tan α − 2 tan β), where 2α is the angle between the rods, and β the angle either rod subtends at the centre.*

Sol. AB and AC are two uniform rods freely jointed at A and resting with their extremeties B and C on the inside of a smooth circular hoop. The radius $OB = r$ of the circular hoop is greater than the length $2a$ of either rod. Let T be the tension in the string connecting the middle points G_1 and G_2 of the rods. The weights W and W of the two rods act at their middle points G_1 and G_2 and the total weight $W + W = 2W$ will act at the middle point M of G_1G_2. Given that $\angle BAL = \angle CAL = \alpha$ and $\angle BOL = \beta$.



Give the system a small displacement in which the angle α changes to $\alpha + \delta\alpha$ and β changes to $\beta + \delta\beta$. The smooth circular hoop remains fixed and hence its centre O can be taken as the fixed point. The lengths of the rods AB and AC do not change while the length of the string G_1G_2 changes.

The equation of virtual work is

$$-T\delta(G_1G_2) + 2W\delta(OM) = 0$$

or $\quad -T\delta(2a\sin\alpha) + 2W\delta(r\cos\beta - a\cos\alpha) = 0$

or $\quad a(-T\cos\alpha + W\sin\alpha)\,\delta\alpha = Wr\sin\beta\,\delta\beta. \qquad \ldots(1)$

In triangle $\quad OBL,\ BL = r\sin\beta,$

and in triangle $\quad ABL,\ BL = 2a\sin\alpha.$

$\therefore \quad 2a\sin\alpha = r\sin\beta$

$\therefore \quad \delta(2a\sin\alpha) = \delta(r\sin\beta)$

or $\quad 2a\cos\alpha\,\delta\alpha = r\cos\beta\,\delta\beta. \qquad \ldots(2)$

Dividing (1) by (2), we get

$$\frac{a(-T\cos\alpha + W\sin\alpha)}{2a\cos\alpha} = \frac{Wr\sin\beta}{r\cos\beta}$$

or $\quad -T + W\tan\alpha = 2W\tan\beta$

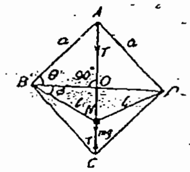
or $\quad T = W(\tan\alpha - 2\tan\beta).$

Ex.34. *A frame, formed of four light rods, each of length a, freely jointed at A, B, C, D is suspended at A. A mass m is suspended from B and D by two strings of length l (l > a/√2). The frame is kept in the form of a square by a string AC. Apply the method of virtual work to find the tension T in AC and show that when*

$$l = a\sqrt{5},\ T = 2mg/3.$$

Sol. The framework is suspended from A and so A is a fixed point from which the distances are to be measured. A mass m is suspended from B and D by means of two strings BN and DN each of length l. Thus a weight mg acts at N. Let T be the tension in the string AC. In the position of equilibrium the figure is a square.

Let $\angle ABD = \theta$ and $\angle NBO = \phi$.



Give the system a small symmetrical displacement about the vertical AC in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA remain fixed and the length AC changes. The lengths of the strings BN and DN remain fixed so that the work done by their tensions is zero. The point N is slightly displaced.

We have

$$AC = 2AO = 2a\sin\theta,$$

and the depth of N below A

$$= AN = AO + ON = a\sin\theta + l\sin\phi.$$

The equation of virtual work is

$$-T\delta(2a\sin\theta) + mg\,\delta(a\sin\theta + l\sin\phi) = 0$$

or $\quad -2aT\cos\theta\,\delta\theta + a\,mg\cos\theta\,\delta\theta + l\,mg\cos\phi\,\delta\phi = 0$

or $\quad a\cos\theta\,(2T - mg)\,\delta\theta = l\,mg\cos\phi\,\delta\phi. \qquad \ldots(1)$

Now from the ΔAOB, $BO = a\cos\theta$

and from the ΔBON, $\quad BO = l\cos\phi.$

$\therefore \quad a\cos\theta = l\cos\phi$

so that $\quad -a\sin\theta\,\delta\theta = -l\sin\phi\,\delta\phi.$

or $\quad a\sin\theta\,\delta\theta = l\sin\phi\,\delta\phi. \qquad \ldots(2)$

Dividing (1) by (2), we have

$$\frac{\cos\theta\,(2T - mg)}{\sin\theta} = \frac{mg\cos\phi}{\sin\phi}$$

or $\quad \cot\theta\,(2T - mg) = mg\cot\phi$

or $\quad 2T - mg = mg\tan\theta\cot\phi$

or $\quad T = \frac{1}{2}mg(1 + \tan\theta\cot\phi).$

In the position of equilibrium $\theta = 45°$,

$$BO = a\cos 45° = a/\sqrt{2},$$

$$ON = \sqrt{(BN^2 - BO^2)} = \sqrt{\{l^2 - (a^2/2)\}}$$

$$= \{\sqrt{(2l^2 - a^2)}\}/\sqrt{2},$$

so that $\quad \cot\phi = \dfrac{BO}{ON} = \dfrac{a/\sqrt{2}}{\{\sqrt{(2l^2 - a^2)}\}/\sqrt{2}}$

$$= \frac{a}{\sqrt{(2l^2 - a^2)}}$$

$$\therefore \quad T = \frac{1}{2}mg\left\{1 + \tan 45°\cdot\frac{a}{\sqrt{(2l^2 - a^2)}}\right\}$$

$$= \frac{1}{2}mg\left\{1 + \frac{a}{\sqrt{(2l^2 - a^2)}}\right\}$$

When, $l = a\sqrt{5}$, the tension T

$$= \frac{1}{2}mg\left\{1 + \frac{a}{\sqrt{(2a^2\cdot 5 - a^2)}}\right\}$$

$$= \frac{1}{2}mg\left(1 + \frac{1}{3}\right) = \frac{2}{3}mg.$$

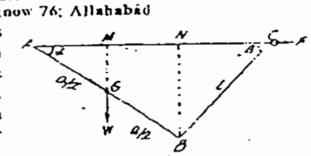
Ex.35 *A rod is movable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is*

$$\frac{W\cos\alpha\cos\beta}{2\sin(\alpha+\beta)},$$

where W is the weight of the rod, and α, β the inclinations of the rod and the string to the horizontal.

[Lucknow 76; Allahabad]

Sol. The rod AB is hinged at A. Let the length of the rod AB be a and the length of the string BC be l. At C there is a ring which can slide on a smooth horizontal wire AC.



Let P be the horizontal force applied at the ring C to keep it at rest. The weight W of the rod AB acts at its middle point G.

Let $\quad \angle BAC = \alpha$ and $\angle BCA = \beta$.

Give the system a small displacement in which α changes to $\alpha + \delta\alpha$ and β changes to $\beta + \delta\beta$. The point A remains fixed. The length of the rod AB remains fixed and the length of the string BC also remains fixed so that the work done by its tension is zero. The points G and C are slightly displaced. We have

$$\text{the depth of } G \text{ below } A = MG$$

$$= AG\sin\alpha = \tfrac{1}{2}a\sin\alpha.$$

and the horizontal distance of C from $A = AC$

$$= AN + NC = a\cos\alpha + l\cos\beta.$$

The equation of virtual work is

$$W\delta(\tfrac{1}{2}a\sin\alpha) + P\delta(a\cos\alpha + l\cos\beta) = 0$$

or $\quad \frac{1}{2}aW\cos\alpha\,\delta\alpha - aP\sin\alpha\,\delta\alpha - lP\sin\beta\,\delta\beta = 0$

or $\quad a(\frac{1}{2}W\cos\alpha - P\sin\alpha)\,\delta\alpha = lP\sin\beta\,\delta\beta. \qquad \ldots(1)$

From the figure, equating the values of BN found from the triangles ANB and CNB, we get

$$a \sin \alpha = l \sin \beta,$$

so that $$a \cos \alpha\, \delta\alpha = l \cos \beta\, \delta\beta. \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{\frac{1}{2}W \cos \alpha - P \sin \alpha}{\cos \alpha} = \frac{P \sin \beta}{\cos \beta}$$

or $$\tfrac{1}{2}W \cos \alpha \cos \beta - P \sin \alpha \cos \beta = P \cos \alpha \sin \beta$$

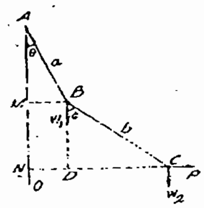
or $$P (\sin \beta \cos \alpha + \cos \beta \sin \alpha) = \tfrac{1}{2} W \cos \alpha \cos \beta$$

or $$P = \frac{W \cos \alpha \cos \beta}{2 \sin (\alpha + \beta)}.$$

Ex.36 *Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then*

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi.$$

Sol. Let the length of the portion AB of the string be a and that of BC be b. The point A is fixed and the vertical line AO through A is a fixed line.



From the fixed point A, the depth of B

$$= AM = a \cos \theta.$$

and the depth of C

$$= AN = AM + MN$$
$$= AM + BD = a \cos \theta + b \cos \phi.$$

Also the horizontal distance of the point C from the fixed line $AO = NC$

$$= ND + DC = MB + DC = a \sin \theta + b \sin \phi.$$

Now give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point A remains fixed, the length of the string remains unaltered and the points B and C are slightly displaced. The equation of virtual work is

$$W_1 \delta (a \cos \theta) + W_2 \delta (a \cos \theta + b \cos \phi) + P\delta (a \sin \theta + b \sin \phi) = 0$$

or $$-aW_1 \sin \theta\, \delta\theta - aW_2 \sin \theta\, \delta\theta - bW_2 \sin \phi\, \delta\phi + aP \cos \theta\, \delta\theta + bP \cos \phi\, \delta\phi = 0$$

or $$a [P \cos \theta - (W_1 + W_2) \sin \theta]\, \delta\theta = b [W_2 \sin \phi - P \cos \phi]\, \delta\phi. \quad \ldots(1)$$

where θ and ϕ are independent of each other.

Now consider a displacement when only θ changes and ϕ does not change so that $\delta\phi = 0$. Then putting $\delta\phi = 0$ in (1), we have

$$a [P \cos \theta - (W_1 + W_2) \sin \theta]\, \delta\theta = 0$$

or $$P \cos \theta - (W_1 + W_2) \sin \theta = 0 \qquad [\because \delta\theta \neq 0]$$

or $$P = (W_1 + W_2) \tan \theta. \quad \ldots(2)$$

Again consider a displacement when only ϕ changes and θ does not change so that $\delta\theta = 0$. Thus putting $\delta\theta = 0$ in (1), we have

$$b [W_2 \sin \phi - P \cos \phi]\, \delta\phi = 0$$

or $$W_2 \sin \phi - P \cos \phi = 0 \qquad [\because \delta\phi \neq 0]$$

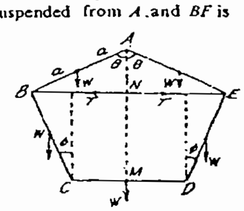
or $$P = W_2 \tan \phi. \quad \ldots(3)$$

From (2) and (3), we have

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi.$$

Ex.37 *Five equal uniform rods, freely jointed at their ends, form a regular pentagon ABCDE and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is W cot $\frac{1}{10}\pi$, where W is the weight of the rod.*

Sol. $ABCDE$ is a pentagon formed of five equal rods each of weight W and say of length $2a$. It is suspended from A and BE is jointed by a weightless bar. Let T be the thrust in the bar BE. The line AM joining A to the middle point M of CD is vertical and the line BE is horizontal. The weights of the rods AB, BC, CD, DE and EA act at their respective middle points. In the position of equilibrium the pentagon is a regular one so that each of its interior angles is $180° - 72°$ *i.e.*, $108°$ or $\frac{3}{5}\pi$ radians.



Let θ be the angle which the two upper slant rods AB and AE make with the vertical and ϕ be the angle which the two lower slant rods CB and DE make with the vertical.

Replace the rod BE by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AM in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$.

The point A remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length BE changes and the middle points of the rods AB, BC etc. are slightly displaced. The $\angle ANB$ remains $90°$.

We have

$$BE = 2BN = 2.2a \sin \theta = 4a \sin \theta,$$

the depth of the middle point of AB or AE below $A = a \cos \theta$,

the depth of the middle point of BC or ED below A

$$= 2a \cos \theta + a \cos \phi,$$

and the depth of the middle point M of CD below A

$$= 2a \cos \theta + 2a \cos \phi.$$

The equation of virtual work is

$$T\delta (4a \sin \theta) + 2W\delta (a \cos \theta) + 2W\delta (2a \cos \theta + a \cos \phi) + W\delta (2a \cos \theta + 2a \cos \phi) = 0$$

or $$4aT \cos \theta\, \delta\theta - 2aW \sin \delta\theta - 4aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi = 0$$

or $$4a (T \cos \theta - 2W \sin \theta)\, \delta\theta = 4aW \sin \phi\, \delta\phi$$

or $$(T \cos \theta - 2W \sin \theta)\, \delta\theta = W \sin \phi\, \delta\phi. \quad \ldots(1)$$

From the figure, finding the length BE in two ways *i.e.*, from the upper portion ABE and from the lower portion $BCDE$, we have

$$4a \sin \theta = 2a + 4a \sin \phi.$$

Differentiating, we get $4a \cos \theta\, \delta\theta = 4a \cos \phi\, \delta\phi$

or $$\cos \theta\, \delta\theta = \cos \phi\, \delta\phi. \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{T \cos \theta - 2W \sin \theta}{\cos \theta} = \frac{W \sin \phi}{\cos \phi}$$

or $$T - 2W \tan \theta = 2W \tan \phi$$

or $$T = W (\tan \phi + 2 \tan \theta).$$

But in the position of equilibrium,

$$\theta = \frac{1}{2} \cdot \frac{3}{5}\pi = \frac{3}{10}\pi,\ \phi = \frac{3}{5}\pi - \frac{1}{2}\pi = \frac{1}{10}\pi.$$

$$\therefore\ T = W\left(\tan \frac{1}{10}\pi + 2 \tan \frac{3}{10}\pi\right) = W\left[\tan \frac{1}{10}\pi + 2 \cot \frac{2}{10}\pi\right]$$

$$\left[\because \tan \frac{3}{10}\pi = \cot\left(\frac{1}{2}\pi - \frac{3}{10}\pi\right) = \cot \frac{2}{10}\pi\right]$$

$$= W\left[\tan \frac{1}{10}\pi + 2 \cdot \frac{1 - \tan^2 (\pi/10)}{2 \tan (\pi/10)}\right]$$

$$\left[\because \cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1 - \tan^2 \alpha}{2 \tan \alpha}\right]$$

$$= W \cot (\pi/10).$$

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Ex.38 *A regular pentagon ABCDE formed of equal uniform rods each of weight W, is suspended from the point A and is maintained in shape by a light rod joining the middle points of BC and DE. Prove that the stress in the light rod is 2W cot (π/10).*

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Sol. Proceed as in part (a).

Ex.39 *A freely jointed framework is formed of five equal uniform rods each of weight W. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles θ and ϕ respectively with the vertical, prove that the tension of the string is to the weight of the rod as*

$$(4 \tan \theta + 2 \tan \phi) : (\tan \theta + \tan \phi).$$

Sol. Draw figure as in Ex. 30 (a). This question differs from the preceding one in having the string AM instead of the rod BE.

Let T be the tension in the string AM. The string AM is given to be inextensible, therefore before giving the displacement replace the string by two equal and opposite forces T so that the length AM may be changed.

Here $AM = 2a \cos \theta + 2a \cos \phi$.

The equation of virtual work is

$$-T\delta (2a \cos \theta + 2a \cos \phi) + 2W\delta (a \cos \theta) + 2W\delta (2a \cos \theta + a \cos \phi) + W\delta (2a \cos \theta + 2a \cos \phi) = 0$$

or $$2aT \sin \theta\, \delta\theta + 2aT \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 4aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 2aW \sin \phi\, \delta\phi = 0$$

or $$2a \sin \theta (T - 4W)\, \delta\theta = 2a \sin \phi (2W - T)\, \delta\phi$$

or $$\sin \theta (T - 4W)\, \delta\theta = \sin \phi (2W - T)\, \delta\phi. \quad \ldots(1)$$

Also from the figure, we have

$$4a \sin \theta = 2a + 4a \sin \phi,$$

so that $$4a \cos \theta\, \delta\theta = 4a \cos \phi\, \delta\phi$$

or $$\cos \theta\, \delta\theta = \cos \phi\, \delta\phi. \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\tan \theta (T - 4W) = \tan \phi (2W - T)$$

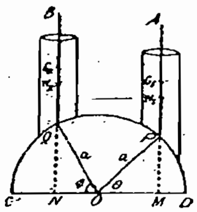
or $$T (\tan \theta + \tan \phi) = W (2 \tan \phi + 4 \tan \theta)$$

or $$\frac{T}{W} = \frac{4 \tan \theta + 2 \tan \phi}{\tan \theta + \tan \phi},$$ which proves the required result.

Ex.40 *A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.*

Sol. Let W_1 and W_2 be the weights of the beams AP and BQ

whose lengths are say $2l_1$ and $2l_2$ respectively. Let θ and ϕ be the angles which the radii OP and OQ make with the horizontal diameter COD of the board. Let a be the radius of the board.



Here COD is a fixed horizontal line. The weight W_1 of the beam AP acts at its centre of gravity G_1 whose height above

$$CD = MG_1 = l_1 + a\sin\theta.$$

The weight W_2 of the beam BQ acts at G_2 whose height above CD is

$$l_2 + a\sin\phi.$$

Let the beams be imagined to undergo a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The equation of virtual work is $-W_1\delta(l_1+a\sin\theta) - W_2\delta(l_2+a\sin\phi)=0$

or $$-aW_1\cos\theta\,\delta\theta = aW_2\cos\phi\,\delta\phi = 0$$

or $$-W_1\cos\theta\,\delta\theta = W_2\cos\phi\,\delta\phi. \quad ...(1)$$

If b be the distance between the tubes in which the beams slide, then from the figure

$$a\cos\theta + a\cos\phi = b = \text{constant}$$

so that, $$-a\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi = 0$$

or $$-\sin\theta\,\delta\theta = \sin\phi\,\delta\phi. \quad ...(2)$$

Dividing (1) by (2), we have

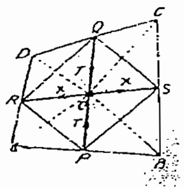
$$W_1\cot\theta = W_2\cot\phi$$

or $$\frac{W_1}{W_2} = \frac{\cot\phi}{\cot\theta} = \frac{\tan\theta}{\tan\phi}$$, which gives the required ratio.

Ex.41 *A smoothly jointed framework of light rods forms a quadrilateral $ABCD$. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other pair by a light rod in a state of thrust X: show, by the method of virtual work, that*

$$T/PQ = X/RS.$$

Sol. $ABCD$ is a framework in the form of a quadrilateral formed of four light rods. The middle points P and Q of the rods AB and DC are joined by a string in a state of tension T and the middle points R and S of the rods AD and BC are joined by a light rod in a state of thrust X. [The framework is to be taken as placed on some smooth horizontal plane].



Since P, S, Q, R are the middle points of the sides of the quadrilateral $ABCD$, therefore $PSQR$ is a parallelogram. Consequently the diagonals PQ and RS of this parallelogram bisect each other at O.

Replace the string PQ by two equal and opposite forces T as shown in the figure and replace the rod RS by two equal and opposite forces X as shown in the figure. Now give the system a small displacement in which PQ changes to $PQ+\delta(PQ)$ and RS changes to $RS+\delta(RS)$. The lengths of the rods AB, BC, CD, DA do not change. The equation of virtual work is

$$-T\delta(PQ) + X\delta(RS) = 0$$

or $$T\delta(PQ) = X\delta(RS)$$

or $$\frac{\delta(PQ)}{\delta(RS)} = \frac{X}{T} \quad ...(1)$$

Now let us find a relation between the parameters PQ and RS from the figure. Since OP is a median of the ΔOAB, therefore

$$OA^2+OB^2 = 2OP^2+2AP^2 = 2(\tfrac{1}{2}PQ)^2+2(\tfrac{1}{2}AB)^2$$
$$=\tfrac{1}{2}(PQ^2+AB^2). \quad ...(2)$$

Similarly from ΔOCD, we have

$$OC^2+OD^2 = \tfrac{1}{2}(PQ^2+CD^2). \quad ...(3)$$

Adding (2) and (3), we get

$$OA^2+OB^2+OC^2+OD^2 = \tfrac{1}{2}(2PQ^2+AB^2+CD^2) \quad ...(4)$$

Doing the same thing with ΔOAD and ΔOBC, we get

$$OA^2+OB^2+OC^2+OD^2 = \tfrac{1}{2}(2RS^2+BC^2+DA^2). \quad ...(5)$$

From (4) and (5), we get

$$\tfrac{1}{2}(2PQ^2+AB^2+CD^2) = \tfrac{1}{2}(2RS^2+BC^2+DA^2)$$

or $$2(PQ^2-RS^2) = BC^2+DA^2-AB^2-CD^2$$

or $$PQ^2-RS^2 = \text{constant}, \quad ...(6)$$

since AB, BC, CD, DA are all of fixed lengths.

Differentiating (6), we get

$$2PQ\,\delta(PQ) - 2RS\,\delta(RS) = 0$$

or $$\frac{\delta(PQ)}{\delta(RS)} = \frac{RS}{PQ} \quad ...(7)$$

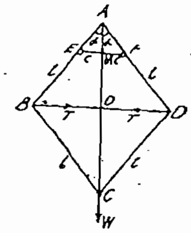
Equating the values of $\dfrac{\delta(PQ)}{\delta(RS)}$ from (1) and (7), we get

$$\frac{X}{T} = \frac{RS}{PQ} \text{ or } \frac{X}{RS} = \frac{T}{PQ}.$$

Ex.42 *$ABCD$ is a rhombus with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD, B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is 2α, show that the tension in the string is*

$$W\tan\alpha\left(\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right)$$

Sol. The rods AB and AD of the frame rest on two smooth pegs E and F which are in the same horizontal line and $EF=2c$. The length of each rod of the rhombus is l and the rods forming the rhombus are weightless. A weight W is attached to the lowest point C. Let T be the tension in the string BD. We have



$$\angle BAC = \alpha = \angle CAD.$$

The diagonal AC is vertical and BD is horizontal.

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$. We have

$$BD = 2BO = 2AB\sin\alpha = 2l\sin\alpha.$$

Also the depth of the point C below EF

$$= MC = AC - AM = 2AO - AM$$
$$= 2AB\cos\alpha - EM\cot\alpha = 2l\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta(2l\sin\alpha) + W\delta(2l\cos\alpha - c\cot\alpha) = 0$$

or $$-2l\,T\cos\alpha\,\delta\alpha - 2lW\sin\alpha\,\delta\alpha + Wc\operatorname{cosec}^2\alpha\,\delta\alpha = 0$$

or $$(-2l\,T\cos\alpha - 2l\,W\sin\alpha + Wc\operatorname{cosec}^2\alpha)\,\delta\alpha = 0$$

or $$-2l\,T\cos\alpha - 2l\,W\sin\alpha + Wc\operatorname{cosec}^2\alpha = 0 \quad [\because \delta\alpha \neq 0]$$

or $$2l\,T\cos\alpha = Wc\operatorname{cosec}^2\alpha - 2l\,W\sin\alpha$$

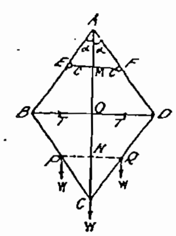
or $$T = \frac{1}{2l\cos\alpha}\{Wc\operatorname{cosec}^2\alpha - 2lW\sin\alpha\}$$

$$= W\tan\alpha\left[\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right]$$

Ex.43 *A rhombus $ABCD$ formed of four weightless rods each of length a freely jointed at the extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant $2c$ apart and in contact with AB and AD. Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If 2α be the angle of the rhombus at A, apply the principle of virtual work to find the tension of the string.*

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and

$$EF = 2c.$$



The length of each rod of the rhombus is a and the rods forming the rhombus are light. Weights each equal to W are hung from the lowest corner C and from the middle points P and Q of the lower sides BC and CD. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the inextensible string joining B and D. We have

$$\angle BAC = \alpha = \angle DAC.$$

Replace the string BD by two equal and opposite forces T as shown in the figure so that the distance BD can be changed. Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$.

We have

$$BD = 2BO = 2AB\sin\alpha = 2a\sin\alpha.$$

The depth of C below $EF = MC = AC - AM$

$$= 2AO - AM = 2AB\cos\alpha - EM\cot\alpha$$
$$= 2a\cos\alpha - c\cot\alpha,$$

and the depth of P or Q below EF

$$= AN - AM = \tfrac{3}{2}AO - AM$$
$$= \tfrac{3}{2}a\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta(2a\sin\alpha) + W\delta(2a\cos\alpha - c\cot\alpha) + 2W\delta(\tfrac{3}{2}a\cos\alpha - c\cot\alpha) = 0$$

or $$(-2aT\cos\alpha - 2aW\sin\alpha + Wc\operatorname{cosec}^2\alpha - 3aW\sin\alpha + 2cW\operatorname{cosec}^2\alpha]\,\delta\alpha = 0$$

or $\quad -2aT\cos\alpha - 5aW\sin\alpha + 3Wc\operatorname{cosec}^2\alpha = 0 \quad [\because \delta\alpha \neq 0]$

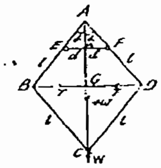
or $\quad 2aT\cos\alpha = 3Wc\operatorname{cosec}^2\alpha - 5aW\sin\alpha$

or $$T = \frac{W(3c\operatorname{cosec}^2\alpha - 5a\sin\alpha)}{2a\cos\alpha}$$

Ex.44 *ABCD is a rhombus formed with four rods each of length l and of weight w joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is 2d and the angle at A is 2α, show that the tension in the string is*

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha - (W+2w)\right].$$

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and $EF=2d$. The length of each rod forming the rhombus is l. The total weight $4w$ of the rods forming the rhombus can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W is attached to the lowest point C. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the string BD. We have



$$\angle BAC = \alpha = \angle DAC.$$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remain fixed and the distances will be measured from this line. The $\angle AGB$ remains $90°$.

We have the length of the string BD
$$=2BG=2AB\sin\alpha=2l\sin\alpha$$
The depth of G below EF
$$=MG=AG-AM=l\cos\alpha-d\cot\alpha,$$
and the depth of C below EF
$$=MC=AC-AM=2l\cos\alpha-d\cot\alpha.$$
The equation of virtual work is
$$-T\delta(2l\sin\alpha)+4w\delta(l\cos\alpha-d\cot\alpha)+W\delta(2l\cos\alpha-d\cot\alpha)=0$$

or $\quad -2lT\cos\alpha\,\delta\alpha - 4lw\sin\alpha\,\delta\alpha + 4dw\operatorname{cosec}^2\alpha\,\delta\alpha - 2lW\sin\alpha\,\delta\alpha + dW\operatorname{cosec}^2\alpha\,\delta\alpha = 0$

or $\quad [-2lT\cos\alpha - 2l\sin\alpha(2w+W) + d\operatorname{cosec}^2\alpha(4w+W)]\delta\alpha = 0$

or $\quad -2lT\cos\alpha - 2l\sin\alpha(2w+W) + d\operatorname{cosec}^2\alpha(4w+W) = 0 \quad [\because \delta\alpha\neq 0]$

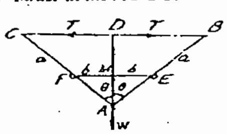
or $\quad 2lT\cos\alpha = d(W+4w)\operatorname{cosec}^2\alpha - 2l\sin\alpha(W+2w)$

or $$T=\frac{1}{2l\cos\alpha}[d(W+4w)\operatorname{cosec}^2\alpha - 2l\sin\alpha(W+2w)]$$

or $$T=\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

Ex.45. *A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant 2b apart. A weight W is suspended from A, find the thrust in the rod BC.*

Sol. ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF=2b$. Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let



$$\angle BAD=\theta=\angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are : (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

We have,
$$BC=2BD=2AB\sin\theta=2a\sin\theta.$$
Also the depth of the point of application A of the weight W below the fixed line EF
$$=MA=ME\cot\theta=b\cot\theta.$$
The equation of virtual work is
$$T\delta(2a\sin\theta)+W\delta(b\cot\theta)=0$$

or $\quad 2aT\cos\theta\,\delta\theta - bW\operatorname{cosec}^2\theta\,\delta\theta=0$

or $\quad (2aT\cos\theta - bW\operatorname{cosec}^2\theta)\,\delta\theta=0$

or $\quad 2aT\cos\theta - bW\operatorname{cosec}^2\theta=0 \quad [\because \delta\theta\neq 0]$

or $\quad 2aT\cos\theta = bW\operatorname{cosec}^2\theta$

or $$T=\frac{Wb}{2a}\operatorname{cosec}^2\theta\sec\theta.$$

But in the position of equilibrium,
$$BC=\tfrac{3}{2}a \text{ and so } BD=\tfrac{3}{4}a.$$

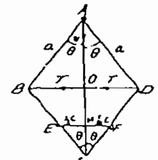
Therefore $$\sin\theta=\frac{BD}{AB}=\frac{\frac{3}{4}a}{a}=\frac{3}{4}$$

and $$\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{(1-\tfrac{9}{16})}=\tfrac{1}{4}\sqrt{7}.$$

$\therefore$ $$T=\frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}}=\frac{32}{9\sqrt{7}}\frac{b}{a}W.$$

Ex.46 *A rhomboidal framework ABCD is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude*
$$W(2a^2c-b^3)/b^2(4a^2-b^2)^{1/2}.$$

Sol. The rods BC and CD of a rhomboidal framework $ABCD$ are in contact with two smooth pegs E and F which are in the same horizontal line and $EF=c$. The rods forming the rhombus are light and the length of each rod forming the rhombus is a. Let T be the thrust in the light rod BD joining B and D. A weight W is placed at the highest joint A. In the position of equilibrium, $BD=b$. The diagonal AC is vertical and BD is horizontal. Let



$$\angle BAC=\theta=\angle CAD.$$

Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The only forces contributing to the sum of virtual works are : (i) the weight W placed at A, and (ii) the thrust T in the rod BD. The reactions of the pegs E and F do not work. We have
$$BD=2BO=2AB\sin\theta=2a\sin\theta.$$
Also the height of A above the fixed line EF
$$=MA=CA-CM$$
$$=2OA-CM=2a\cos\theta-\tfrac{1}{2}c\cot\theta.$$
The equation of virtual work is
$$T\delta(2a\sin\theta)-W\delta(2a\cos\theta-\tfrac{1}{2}c\cot\theta)=0$$

or $\quad 2aT\cos\theta\,\delta\theta+2aW\sin\theta\,\delta\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\theta\,\delta\theta=0$

or $\quad (2aT\cos\theta+2aW\sin\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\theta)\,\delta\theta=0$

or $\quad 2aT\cos\theta+2aW\sin\theta-\tfrac{1}{2}cW\operatorname{cosec}^2\theta=0 \quad [\because \delta\theta\neq 0]$

or $\quad 2aT\cos\theta=\tfrac{1}{2}cW\operatorname{cosec}^2\theta-2aW\sin\theta$

or $$T=W.\frac{\frac{1}{2}c\operatorname{cosec}^2\theta-2a\sin\theta}{2a\cos\theta} \quad \ldots(1)$$

But in the position of equilibrium, we have
$$BD=b \text{ so that } BO=\tfrac{1}{2}b.$$

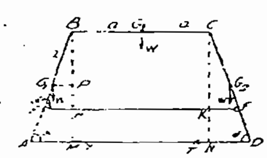
$\therefore$ from $\triangle AOB$, we have
$$\sin\theta=\frac{BO}{AB}=\frac{\frac{1}{2}b}{a}=\frac{b}{2a}.$$

$\therefore$ $\operatorname{cosec}\theta=\dfrac{2a}{b}$ and $\cos\theta=\sqrt{(1-\sin^2\theta)}$
$$=\sqrt{\left(1-\frac{b^2}{4a^2}\right)}=\frac{\sqrt{(4a^2-b^2)}}{2a}.$$

Substituting in (1), we have
$$T=W\frac{\frac{1}{2}c.(4a^2/b^2)-2a.(b/2a)}{2a.\{\sqrt{(4a^2-b^2)}/2a\}}=W.\frac{2a^2c-b^3}{b^2(4a^2-b^2)^{1/2}}.$$

Ex.47 *Three rigid rods AB, BC, CD each of length 2a, are smoothly jointed at B and C. The system is placed in a vertical plane so that rods AB, CD are in contact with two smooth pegs distant 2c apart in the same horizontal line, the rods AB, CD make equal angle α with the horizon. Prove that the tension of the string AD which will maintain this configuration is*
$$\tfrac{1}{2}W\operatorname{cosec}\alpha\sec^2\alpha\,\{(3c/a)-(3+2\cos^3\alpha)\},$$
where W is the weight of either rod.

Sol. Three rigid rods AB, BC, CD, each of length $2a$ and weight W are smoothly jointed at B and C. The rods AB and CD are in contact with two smooth pegs E and F which are in the same horizontal line and
$$EF=2c.$$



Let T be the tension in the string AD joining A and D. The weights W of the rods AB, BC and CD act at their respective middle points.

We have $\quad \angle BAD=\alpha=\angle CDA.$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD do not change and the length AD changes.

We have,
$$AD=AM+MN+ND$$
$$=2a\cos\alpha+2a+2a\cos\alpha$$
$$=4a\cos\alpha+2a.$$
The height of G_1 or G_3 above the fixed line EF
$$=HP=HB-PB=EH\tan\alpha-BG_1\sin\alpha$$
$$=\tfrac{1}{2}(2c-2a)\tan\alpha-a\sin\alpha$$
$$=(c-a)\tan\alpha-a\sin\alpha,$$
and the height of G_2 above EF
$$=HB=(c-a)\tan\alpha.$$

The equation of virtual work is

$$-T\delta\,(4a\cos\alpha+2a)-2W\delta\,\{(c-a)\tan\alpha-a\sin\alpha\}-W\delta\,\{(c-a)\tan\alpha\}=0$$

or $$4aT\sin\alpha\,\delta\alpha-2(c-a)W\sec^2\alpha\,\delta\alpha+2aW\cos\alpha\,\delta\alpha-(c-a)W\sec^2\alpha\,\delta\alpha=0$$

or $$[4aT\sin\alpha-3(c-a)W\sec^2\alpha+2aW\cos\alpha]\,\delta\alpha=0$$

or $$4aT\sin\alpha-3(c-a)W\sec^2\alpha+2aW\cos\alpha=0 \qquad [\because \delta\alpha\neq 0]$$

or $$4aT\sin\alpha=3(c-a)W\sec^2\alpha-2aW\cos\alpha.$$

or $$T=\frac{1}{4a\sin\alpha}W\{3c\sec^2\alpha-3a\sec^2\alpha-2a\cos\alpha\}$$

$$=\tfrac{1}{4}W\operatorname{cosec}\alpha\sec^2\alpha\,[(3c/a)-(3+2\cos^3\alpha)].$$

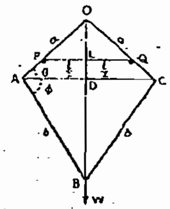
Ex.48 *Four light rods are jointed together to form a quadrilateral $OABC$. The lengths are such that*

$$OA=OC=a,\ AB=CB=b.$$

The framework hangs in a vertical plane OA and OC resting in contact with two smooth pegs distant l apart and on the same horizontal level. A weight hangs at B. If θ,ϕ are the inclinations of OA, AB to the horizontal, prove that these values are given by the equations

$$a\cos\theta=b\cos\phi$$

and $$\tfrac{1}{2}l\sec^2\theta\sin\phi=a\sin(\theta+\phi).$$



Sol. $OABC$ is a framework formed of four light rods such that

$$OA=OC=a \text{ and } AB=CB=b.$$

The rods OA and OC are in contact with two smooth pegs P and Q which are in the same horizontal line and $PQ=l$. A weight W hangs at B. We have

$$\angle OAC=\theta \text{ and } \angle BAC=\phi.$$

Give the system a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W acting at B.

We have, the depth of B below $PQ=LB=OB-OL$

$$=OD+DB-OL=a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta.$$

The equation of virtual work is

$$W\delta\,(a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta)=0$$

or $$a\cos\theta\,\delta\theta+b\cos\phi\,\delta\phi-\tfrac{1}{2}l\sec^2\theta\,\delta\theta=0$$

or $$(\tfrac{1}{2}l\sec^2\theta-a\cos\theta)\,\delta\theta=b\cos\phi\,\delta\phi. \qquad \ldots(1)$$

Now let us find a relation between the parameters θ and ϕ from the figure. From the $\triangle OAD$, we have $AD=a\cos\theta$ and from the $\triangle BAD$, we have $AD=b\cos\phi$.

$$\therefore\quad a\cos\theta=b\cos\phi. \qquad \ldots(2)$$

Differentiating (2).

$$-a\sin\theta\,\delta\theta=-b\sin\phi\,\delta\phi$$

or $$a\sin\theta\,\delta\theta=b\sin\phi\,\delta\phi. \qquad \ldots(3)$$

Dividing (1) by (3), we get

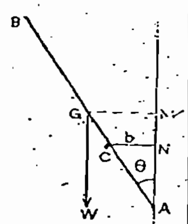
$$\frac{\tfrac{1}{2}l\sec^2\theta-a\cos\theta}{a\sin\theta}=\frac{b\cos\phi}{b\sin\phi}$$

or $$\tfrac{1}{2}l\sec^2\theta\sin\phi=a(\sin\theta\cos\phi+\cos\theta\sin\phi)$$

or $$\tfrac{1}{2}l\sec^2\theta\sin\phi=a\sin(\theta+\phi). \qquad \ldots(4)$$

Thus θ and ϕ are given by the equations (2) and (4).

Ex.49 *A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$.*



Sol. A uniform beam AB of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b. Suppose the rod makes an angle θ with the wall i.e., $\angle BAM=\theta$. The weight W of the rod acts at its middle point G.

Give the rod a small displacement in which θ changes to $\theta+\delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and C do not work.

We have, the height of G above the fixed point C

$$=NM=AM-AN=AG\cos\theta-CN\cot\theta$$
$$=a\cos\theta-b\cot\theta.$$

The equation of virtual work is

$$-W\delta\,(a\cos\theta-b\cot\theta)=0,$$

or $$\delta\,(a\cos\theta-b\cot\theta)=0$$

or $$-a\sin\theta\,\delta\theta+b\operatorname{cosec}^2\theta\,\delta\theta=0$$

or $$(-a\sin\theta+b\operatorname{cosec}^2\theta)\,\delta\theta=0$$

or $$-a\sin\theta+b\operatorname{cosec}^2\theta=0 \qquad [\because \delta\theta\neq 0]$$

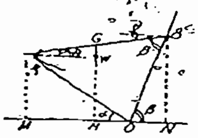
or $$a\sin\theta=b\operatorname{cosec}^2\theta \text{ or } \sin^3\theta=b/a$$

or $$\theta=\sin^{-1}(b/a)^{1/3},$$

giving the inclination of the rod to the vertical in the position of equilibrium.

Ex.50 *A heavy uniform rod, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that* IAS-2007

$$\tan\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta).$$



Sol. Let AB be the rod of length $2a$ and G its middle point. Let AM, BN, and GH be the perpendiculars from A, B and G on the horizontal line through O, the point of intersection of the inclined planes OA and OB. The weight W of the rod acts at G and in equilibrium the rod makes an angle θ with the horizontal.

Give the rod a small displacement in which θ changes to $\theta+\delta\theta$. The horizontal line MON through O is the fixed line from which the distances will be measured. The angles α and β remain fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and B do no work. We have

the height of G above the fixed line MON

$$=HG=\tfrac{1}{2}(AM+BN)=\tfrac{1}{2}(OA\sin\alpha+OB\sin\beta).$$

From the $\triangle AOB$ by the sine theorem of trigonometry, we have

$$\frac{OA}{\sin(\beta-\theta)}=\frac{OB}{\sin(\theta+\alpha)}=\frac{AB}{\sin\{\pi-(\alpha+\beta)\}}=\frac{2a}{\sin(\alpha+\beta)}.$$

$$\therefore\quad OA=2a\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)},\ OB=\frac{2a\sin(\theta+\alpha)}{\sin(\alpha+\beta)}.$$

$$\therefore\quad HG=\tfrac{1}{2}\cdot\frac{2a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}.$$

The equation of virtual work is

$$-W\delta\,(HG)=0, \text{ or } \delta\,(HG)=0$$

or $$\delta\left[\frac{a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}\right]=0$$

or $$\frac{a}{\sin(\alpha+\beta)}[-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta]\,\delta\theta=0$$

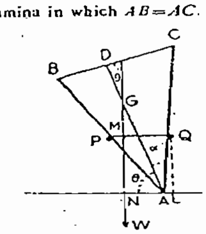
or $$-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta=0 \qquad [\because \delta\theta\neq 0]$$

or $$-(\cos\beta\cos\theta+\sin\beta\sin\theta)\sin\alpha+(\cos\theta\cos\alpha-\sin\theta\sin\alpha)\sin\beta=0$$

or $$2\sin\alpha\sin\beta\sin\theta=\cos\theta(\cos\alpha\sin\beta-\cos\beta\sin\alpha)$$

or $\tan\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta)$, giving the inclination of the rod to the horizontal in the position of equilibrium.

Ex.51 *An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.*



Sol. ABC is an isosceles triangular lamina in which $AB=AC$. The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.

Let $PQ=a$ so that $BC=3a$.

If D is the middle point of BC, then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG=\tfrac{2}{3}AD.$$

The weight W of the lamina acts vertically downwards at G. We have

$$\angle BAD=\angle CAD=\alpha.$$

Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendicular lines, therefore $\angle DAN=\theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line NMG].

Now $$\angle QPA=\angle PAN=\theta-\alpha,$$

and $$\angle QAL=\pi-(\theta+\alpha).$$

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=MG=NG-NM=NG-LQ$$
$$=AG\sin\theta-AQ\sin\{\pi-(\theta+\alpha)\}$$
$$=\tfrac{2}{3}AD\sin\theta-AQ\sin(\theta+\alpha).$$

Now $AD=CD\cot\alpha=\tfrac{3}{2}a\cot\alpha$. Also from the $\triangle AQP$, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ}=\frac{PQ}{\sin PAQ} \text{ i.e., } \frac{AQ}{\sin(\theta-\alpha)}=\frac{a}{\sin 2\alpha}.$$

$$\therefore\quad AQ=\frac{a}{\sin 2\alpha}\sin(\theta-\alpha).$$

$$\therefore\quad MG=\tfrac{2}{3}\cdot\tfrac{3}{2}a\cot\alpha\sin\theta-\frac{a}{\sin 2\alpha}\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{2\sin 2\alpha}2\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{4\sin\alpha\cos\alpha}(\cos 2\alpha-\cos 2\theta)$$

$$=a\cot\alpha\sin\theta-\frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha}+\frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}$$

The equation of virtual work is

$$-W\delta\,(MG)=0,\quad\text{or}\quad \delta\,(MG)=0$$

or $$\delta\left[a\cot\alpha\sin\theta-\frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha}+\frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}\right]=0$$

or $$\left[a\cot\alpha\cos\theta-\frac{2a\sin 2\theta}{4\sin\alpha\cos\alpha}\right]\delta\theta=0$$

or $$a\cot\alpha\cos\theta-\frac{4a\sin\theta\cos\theta}{4\sin\alpha\cos\alpha}=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$a\cos\theta\left(\cot\alpha-\frac{\sin\theta}{\sin\alpha\cos\alpha}\right)=0.$$

$\therefore$ $\cos\theta=0$ *i.e.*, $\theta=\frac{\pi}{2}$, giving one position of equilibrium in which the lamina rests symmetrically on the pegs

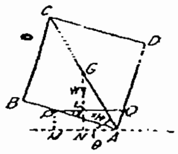
or $\cot\alpha-\dfrac{\sin\theta}{\sin\alpha\cos\alpha}=0$ *i.e.*, $\sin\theta=\cos^2\alpha$ *i.e.*, $\theta=\sin^{-1}(\cos^2\alpha)$,

giving the other position of equilibrium.

Ex.52 *A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4}\ \text{or}\ \frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right).$$

Sol. The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ=c$ and $AB=2a$.



The weight W of the lamina acts at G, the middle point of the diaganal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

$$\angle PAM=\theta=\angle QPA.$$

We have $\angle BAC=\frac{1}{4}\pi=$constant.

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=LG=NG-NL=NG-MP$$
$$=AG\sin(\tfrac{1}{4}\pi+\theta)-AP\sin\theta$$
$$=a\sqrt{2}\sin(\tfrac{1}{4}\pi+\theta)-PQ\cos\theta\sin\theta$$
$$[\because\ AG=\tfrac{1}{2}AC=\tfrac{1}{2}\,2a\sqrt{2}=a\sqrt{2},\ \text{and}\ AP=PQ\cos\theta]$$
$$=a\sqrt{2}(\sin\tfrac{1}{4}\pi\cos\theta+\cos\tfrac{1}{4}\pi\sin\theta)-c\cos\theta\sin\theta$$
$$=a(\cos\theta+\sin\theta)-c\cos\theta\sin\theta.$$

The equation of virtual work is

$$-W\delta\,(LG)=0,\quad\text{or}\quad \delta\,(LG)=0$$

or $$\delta\,[a(\cos\theta+\sin\theta)-c\cos\theta\sin\theta]=0$$

or $$[a(-\sin\theta+\cos\theta)-c(\cos^2\theta-\sin^2\theta)]\,\delta\theta=0$$

or $$a(\cos\theta-\sin\theta)-c(\cos^2\theta-\sin^2\theta)=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$(\cos\theta-\sin\theta)\,[a-c(\cos\theta+\sin\theta)]=0.$$

$\therefore$ either $\cos\theta-\sin\theta=0$

i.e., $\sin\theta=\cos\theta$ *i.e.*, $\tan\theta=1$ *i.e.*, $\theta=\frac{1}{4}\pi$,

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $$a-c(\cos\theta+\sin\theta)=0$$

i.e., $$c^2(\cos\theta+\sin\theta)^2=a^2$$

i.e., $$c^2(1+\sin 2\theta)=a^2$$

i.e., $$\sin 2\theta=\frac{a^2}{c^2}-1=\frac{a^2-c^2}{c^2}$$

i.e., $$\theta=\frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right),$$

giving the other position of equilibrium.

Ex.53 *A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove by the principle of virtual work that the side of length a makes with the vertical an angle θ given by $2c\cos 2\theta=b\cos\theta-a\sin\theta$.*

Sol. Proceed as in part (a).

Ex.54 *Two equal rods, AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then*

$$b\sin^3\theta=a\cos\theta.$$

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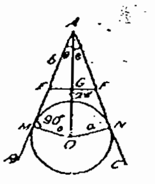


Sol. Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC. If E and F are the middle points of AB and AC, then the total weight $2W$ of the two rods can be taken acting at G, the middle point of EF. The line AO is vertical. We have

$$\angle BAO=\angle CAO=\theta.$$

Also $AB=2b$, $AE=b$. If the rod AB touches the circle at M, then $\angle OMA=90°$ and $OM=$the radius of the circle $=a$.

Give the rods a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90°. We have,

the height of G above the fixed point O

$$=OG=OA-GA=OM\operatorname{cosec}\theta-AE\cos\theta$$
$$=a\operatorname{cosec}\theta-b\cos\theta.$$

The equation of virtual work is

$$-2W\delta\,(OG)=0,\quad\text{or}\quad \delta\,(OG)=0$$

or $$\delta\,(a\operatorname{cosec}\theta-b\cos\theta)=0$$

or $$(-a\operatorname{cosec}\theta\cot\theta+b\sin\theta)\,\delta\theta=0$$

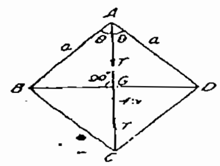
or $$-a\operatorname{cosec}\theta\cot\theta+b\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$a\operatorname{cosec}\theta\cot\theta=b\sin\theta$$

or $$a\cos\theta=b\sin^3\theta.$$

Problems involving elastic strings

Ex.55 *Four equal jointed rods, each of length a are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is $a\sqrt{2}/3$.*



Sol. $ABCD$ is a framework formed of four equal rods each of length a and say of weight W. It is suspended from the point A. A and C are connected by an elastic string and in equilibrium $ABCD$ is square. The diagonal AC is vertical and so BD is horizontal. Let T be the tension in the string AC. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. Let $\angle BAC=\angle DAC$.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed, the length AC changes, the point G is slightly displaced, the lengths of the rods AB, BC, CD, DA do not change, and the $\angle BGA$ remains 90°. We have $AC=2AG=2a\cos\theta$.

Also the depth of G below $A=AG=a\cos\theta$.

The equation of virtual work is

$$-T\delta\,(2a\cos\theta)+4W\delta\,(a\cos\theta)=0$$

or $$2aT\sin\theta\ \delta\theta-4aW\sin\theta\,\delta\theta=0$$

or $$2a\sin\theta\,(T-2W)\,\delta\theta=0$$

or $$T-2W=0 \qquad [\because\ \delta\theta\neq 0\ \text{and}\ \sin\theta\neq 0]$$

or $$T=2W.$$

Let l be the natural length of the elastic string AC. In the position of equilibrium, $\angle BAC=45°$ and so the extended length AC of the elastic string $=2AG=2a\cos 45°=2a/\sqrt{2}=a\sqrt{2}$.

By Hooke's law, the tension T in the elastic string AC is given by $T=\lambda\dfrac{AC-l}{l}$, where λ is the modulus of elasticity of the string

$$=W\frac{a\sqrt{2}-l}{l} \qquad [\because\ \lambda=W]$$

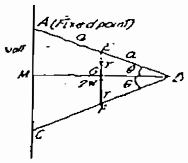
Equating the two values of T, we get

$$2W=W\frac{a\sqrt{2}-l}{l}.$$

or $$2l=a\sqrt{2}-l,\ \text{or}\ 3l=a\sqrt{2}$$

or $$l=a\sqrt{2}/3.$$

Ex.56 *One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2\sin^{-1}(3/4)$.*



Sol. AB and BC are two rods each of length $2a$ and weight W smoothly joined together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a.

Let T be the tension in the string EF.

The total weight $2W$ of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M. Let $\angle ABM=\theta=\angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes to $\theta+\delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF=2EG=2EB\sin\theta=2a\sin\theta$.

Also the depth of G below the fixed point A

$$=AM=AB\sin\theta=2a\sin\theta.$$

The equation of virtual work is

$$-T\delta\,(2a\sin\theta)+2W\delta\,(2a\sin\theta)=0$$

or $$(-2aT\cos\theta+4aW\cos\theta)\,\delta\theta=0$$

or $$2a\cos\theta\,(-T+2W)\,\delta\theta=0$$

or $$-T+2W=0 \qquad [\because\ \delta\theta\neq 0\ \text{and}\ \cos\theta\neq 0]$$

or $$T=2W.$$

Also by Hooke's law the tension T in the elastic string EF is given by

$$T=\lambda\frac{2a\sin\theta-a}{a},$$

where λ is the modulus of elasticity of the string

$$=4W(2\sin\theta-1). \qquad [\because\ \lambda=4W]$$

Equating the two values of T, we have

$$2W = 4W(2\sin\theta - 1)$$

or $\quad 1 = 2(2\sin\theta - 1)$, or $1 = 4\sin\theta - 2$

or $\quad 4\sin\theta = 3$, or $\sin\theta = 3/4$, or $\theta = \sin^{-1}(3/4)$.

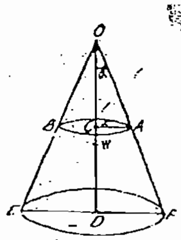
$\therefore$ in equilibrium the whole angle between AB and BC

$$= 2\theta = 2\sin^{-1}(3/4).$$

Ex.57 *A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is*

$$a\left(1 + \frac{W}{2\lambda\pi}\cot\alpha\right).$$

Sol. OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical. A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x. The weight W of the string acts at its centre of gravity C. Let T be the tension in this string.



Give the string a small displacement in which x changes to $x + \delta x$. The point O remains fixed, the point C is slightly displaced, $\angle\alpha$ is fixed and the length of the string slightly changes.

We have the length of the string AB in the form of a circle of radius $x = 2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.

Also the depth of the point of application C of the weight W below the fixed point O

$$= OC = AC\cot\alpha = x\cot\alpha$$

and so the work done by the weight W during this small displacement $= W\delta(x\cot\alpha)$.

Since the reactions at the various points of contact do no work, we have, by the principle of virtual work

$$-T\delta(2\pi x) + W\delta(x\cot\alpha) = 0$$

or $\quad -2\pi T\,\delta x + W\cot\alpha\,\delta x = 0$ or $(-2\pi T + W\cot\alpha)\delta x = 0$

or $\quad -2\pi T + W\cot\alpha = 0 \qquad [\because \delta x \neq 0]$

or $\quad T = (W\cot\alpha)/2\pi.$

By Hooke's law the tension T in the elastic string AB is given by

$$T = \lambda\frac{2\pi x - 2\pi a}{2\pi a} = \lambda\frac{x - a}{a}$$

Equating the two values of T, we get

$$\frac{W\cot\alpha}{2\pi} = \lambda\frac{x - a}{a}$$

or $$x - a = \frac{a}{2\pi\lambda}W\cot\alpha$$

or $$x = a\left(1 + \frac{W}{2\pi\lambda}\cot\alpha\right).$$

which gives the radius of the string in equilibrium.

Ex.58. *An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the verticle angle of the cone to be 2α.*

Sol. Proceed as in Ex. 54. Here in place of a heavy elastic string of weight W we have a heavy endless chain of weight W. If T is the tension in this chain, then proceeding as in Ex. 54, we get $\quad T = (W\cot\alpha)/2\pi$.

* STRINGS IN TWO DIMENSIONS *

Flexible string :- All these strings which offer no resistance on bending at any point are called flexible string. Here, the resultant action across any section of the string consists of a single force, whose line of action is along the tangent to the curve formed by the string.

The normal section of the string is taken to be so small that it may be regarded as a curved line.

The catenary

When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called catenary.

Uniform/Common Catenary

If the weight per unit length of the suspended flexible string/chain is constant, then the catenary is called the uniform/common catenary.

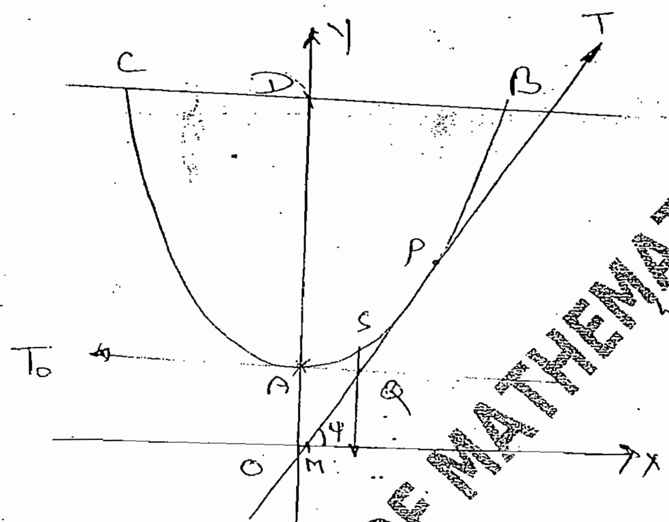
→ Here the word catenary will always mean the common catenary.

*Intrinsic Equation of the Common Catenary

Let the uniform flexible string BAC hang in the form of a uniform catenary with A as its lowest point. Let P be any point on the portion AB of the string and

s = Distance of P from A measured along the arc length of the string.

w = weight per unit length of the string.



∴ Weight of the portion AP = ws

~~The portion AP of the string~~ is in ⇌ under the action of the following three forces

(i) the weight 'ws' of the string AP acting vertically downward through its C.G.

(ii) Tension T_0, at the lowest point A acting along the tangent to the curve at A, which is horizontal

(iii) Tension T, at P, acting along the tangent to the curve at P, inclined at an angle ψ to the horizontal

Since the string AP is in $\rightleftharpoons$ under the action of three forces, acting in the same vertical plane
∴ the line of action of the weight 'ws' must pass through Q, which is the point of intersection of the lines of action of the tension T_0 and T.

Now, for $\rightleftharpoons$

$$\Sigma F_x = 0$$

$$T_0 = T \cos\psi \quad -(1)$$

and $\Sigma F_y = 0$

$$T \sin\psi = ws \quad -(2)$$

∴ Dividing (2) by (1), we have,

$$\tan\psi = \frac{ws}{T_0} \quad -(3)$$

let $T_0 = wc$ = weight of the length 'c' of the string

∴ $\tan\psi = \frac{s}{c}$

$$\Rightarrow \quad s = c\tan\psi \quad -(4)$$

which is the intrinsic equation of the common catenary.

* $\because T_0 = T\cos\varphi$ i.e. the horizontal component of the tension at every point of the catenary is the same and is equal to T_0 (the tension at the lowest point)

* $T\sin\varphi = ws$ i.e. the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and the point.

Cartesian Equation of the common catenary

The intrinsic equation of the common catenary

$$S = c\tan\varphi \quad \text{---(i)}$$

We know that $\frac{dy}{dx} = \tan\varphi$

$$s = c\frac{dy}{dx}$$

Differentiating both side w.r.t. 'x' we have.

$$\frac{ds}{dx} = c\frac{d^2y}{dx^2}$$

$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2} = c\frac{d^2y}{dx^2}$$

put, $\frac{dy}{dx} = P \Rightarrow \frac{d^2y}{dx^2} = \frac{dP}{dx}$, we have

$$\Rightarrow \sqrt{1+p^2} = c\frac{d^2y}{dx^2} = c\frac{dp}{dx}$$

or, $$\frac{dx}{c} = \frac{dp}{\sqrt{1+p^2}}$$

Integrating, we have,

$$\frac{x}{c} + A = \sinh^{-1}(p)$$

$$= \sinh^{-1}\left(\frac{dy}{dx}\right) \quad \text{--- (2)}$$

where, A = constant of integration.

If we choose the vertical line through the lowest point 'A' of the catenary as the axis of y, then at point A, we have,

$$x = 0 \text{ and, } \frac{dy}{dx} = 0$$

from (2), $A = 0$

$$\therefore \frac{x}{c} = \sinh^{-1}\left(\frac{dy}{dx}\right)$$

or, $$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right)$$

Now, integrating both sides w.r.t. 'x' we have,

$$y = c\cosh\left(\frac{x}{c}\right) + B \quad \text{--- (3)}$$

where, B = constant of integration.

If we take the origin 'O' at a depth 'c' below the lowest point A of the catenary then at A, we have,

$x = 0, \; y = c$

$\therefore$ from (3), we have,

$B = 0$

$\therefore y = c \cosh\left(\frac{x}{c}\right)$ — (4)

which is the Cartesian equation of the common catenary.

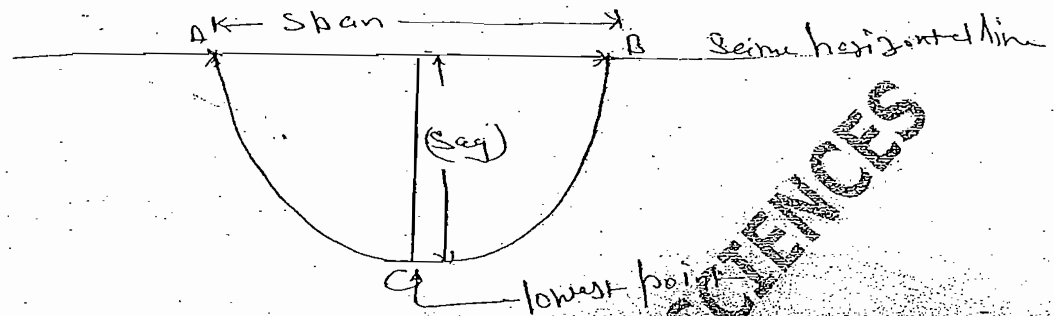
(1) Axis of the catenary: Since, $\cosh\left(\frac{x}{c}\right)$ is an even function of x, therefore the curve is symmetrical about the axis of which is along the vertical through the lowest point of catenary. This vertical line of symmetry is called the axis of catenary.

(2) Vertex of the catenary: The lowest point 'A' of the common catenary at which the tangent is horizontal is called the vertex of the catenary.

(3) Parameter of the catenary: c of $y = c\cosh\left(\frac{x}{c}\right)$

(4) Directrix of the catenary: the horizontal line at a depth 'c' below the lowest point i.e. x axis is called the directrix of the catenary.

(5) Span and Sag



* Relation between x and s

$\because \quad s = c\tan\psi = c\frac{dy}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{s}{c} \quad \text{—} (1)$

Also, $\quad y = c\cosh\left(\frac{x}{c}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \sinh\left(x/c\right) \quad \text{—} (2)$

from (1) and (2), we have,

$\frac{s}{c} = \sinh\left(x/c\right)$

or, $\quad s = c\sinh\left(x/c\right)$

which is the relation between x and s.

* Relation between y and s

$\because \; y = c\cosh\left(x/c\right)$

and $\quad s = c\sinh\left(x/c\right)$

Squaring and we have,

$$\therefore\ y^2 - s^2 = c^2\left(\cosh^2(x/c) - \sinh^2(x/c)\right)$$

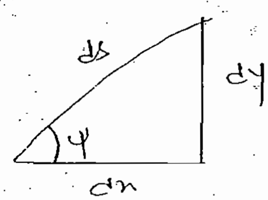
$$\Rightarrow y^2 - s^2 = c^2$$

or, $y^2 = c^2 + s^2$

Which is the relation between y and s.

* Relation between y and ψ

For any curve, we have,



$$\frac{dy}{ds} = \sin\psi$$

$$\therefore\ \frac{dy}{d\psi} = \frac{dy}{ds}\cdot\frac{ds}{d\psi}$$

$$= \sin\psi \cdot \frac{d}{d\psi}(c\cdot\tan\psi)$$

$$= c\sin\psi\cdot\sec^2\psi$$

$$\therefore\ \frac{dy}{d\psi} = c\sec\psi\cdot\tan\psi$$

Integrating, we get,

$$y = c\sec\psi + A$$

where, A = Constant of integration.

When, $y = c$, $\psi = 0$, $\therefore A = 0$

$\therefore\ y = c\sec\varphi$

Which is the relation b/n y and φ.

* Relation b/n x and φ

$\because$ for any curve,

$$\frac{dx}{ds} = \cos\varphi$$

or, $$\frac{dx}{d\varphi} = \frac{dx}{ds}\cdot\frac{ds}{d\varphi}$$

$$= \cos\varphi\cdot\frac{d}{d\varphi}(c\tan\varphi) \quad [\because s = c\tan\varphi]$$

or, $$\frac{dx}{d\varphi} = \cos\varphi\, c.\sec^2\varphi$$

or, $$\frac{dx}{d\varphi} = c.\sec\varphi$$

$$\Rightarrow x = c.\log(\sec\varphi + \tan\varphi) + B$$

Where, B is a constant of integration.

But when $x=0$, $\varphi=0$ $\therefore B=0$

$$\therefore x = c.\log(\sec\varphi + \tan\varphi)$$

* Relation between tension and ordinate

∵ we have,

$$T\cos\psi = T_0 \text{ and } T_0 = wc$$

$$\therefore T = T_0 \sec\psi = wc\sec\psi$$

But, $y = c\sec\psi$

$$\therefore \boxed{T = wy}$$

This shows that the tension at any point of a catenary varies as the height of the point above the directrix.

* Radius of curvature at any point of a catenary

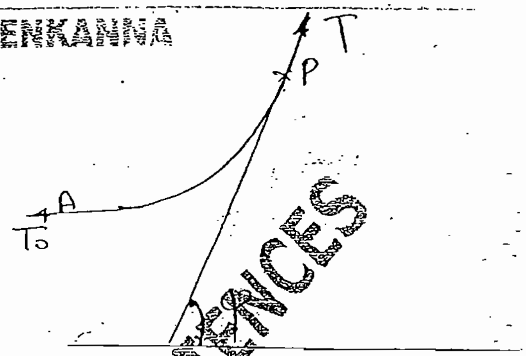
we have, $s = c\tan\psi$

$$\rho = \frac{ds}{d\psi} = c\sec^2\psi.$$

Q.1. (IAS-2000) If T be the tension at any point P of a catenary and T_0 that at the lowest point A, prove that

$$T^2 - T_0^2 = W^2$$

where, W being the weight of the arc AP of the catenary.



Soln :

Let $AP = s$ and

φ be the inclination of the tangent at P to the horizontal.

If w = weight per unit length.

$\therefore W = ws$

Let T = tension at point 'P' of catenary

T_0 = tension at the lowest point.

$\therefore T\cos\varphi = T_0$

and $T\sin\varphi = ws = W$

Squaring and adding, we have,

$$T^2 = T_0^2 + W^2$$

$$\Rightarrow T^2 - T_0^2 = W^2$$

Proved

Q.2 Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their ~~weights~~ heights

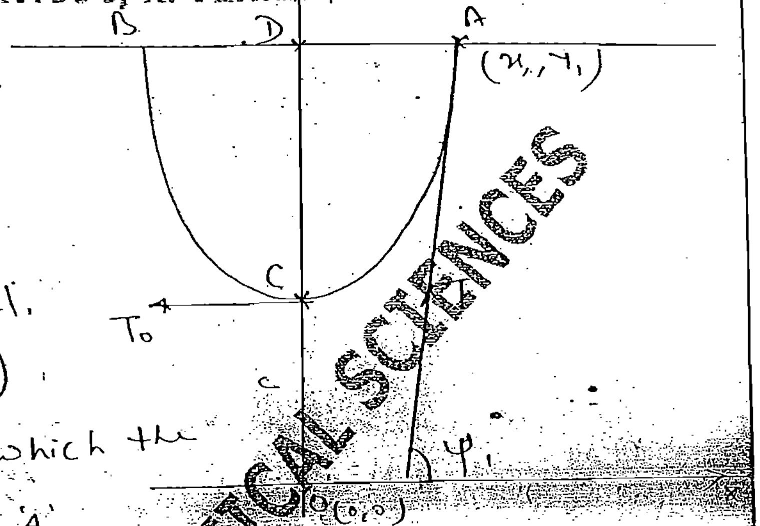
Sol: $\because T \propto y$

Q.3. A rope of length 2l feet is suspended between two points at the same level and the lowest point of the rope is 'b' feet below the point of suspension. Show that the horizontal component of the tension is $\frac{w(l^2-b^2)}{2b}$, w being the weight of the rope per foot of its length.

Q.4. A uniform chain of length l is to be suspended from two points A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{n^2-1}} \log\left(n+\sqrt{n^2-1}\right)$$

Sol'n:



Let the uniform chain ACB of length 'l' be suspended from two points A and B in the same horizontal level.

Let point A = (x_1, y_1)

and φ_1 = angle at which the tangent at 'A' makes with the x-axis

T = tension at point A

T_0 = ~~tension at the~~ lowest point C

∴ Given that $T = n T_0$ — (1)

But $T = w y_1$ and $T_0 = wc$

[where, w = weight per unit length]

∴ $w y_1 = nwc$

$\Rightarrow w \cdot c \sec\varphi_1 = nwc$ $[\because y = c\sec\varphi]$

$\Rightarrow \sec\varphi_1 = n$ — (2)

Now, for the point A,

s = arc length CA = $\frac{l}{2}$

from, $s = c\tan\varphi$, we have

$\frac{1}{2} = c \tan\psi_1$

$\Rightarrow c = \frac{1}{2\tan\psi_1} = \frac{1}{2\sqrt{\sec^2\psi_1 - 1}}$

$\Rightarrow c = \frac{1}{2\sqrt{n^2-1}}$

and,

$\therefore x = c\log(\sec\psi_1 + \tan\psi_1)$

$= \frac{1}{2\sqrt{n^2-1}} \cdot \log(\sec\psi_1 + \tan\psi_1)$

$= \frac{1}{2\sqrt{n^2-1}} \log\left(n + \sqrt{n^2-1}\right)$

$[\because \sec\psi_1 = n]$

Now the span $AB = 2AD$

$= 2x$

$= \frac{1}{\sqrt{n^2-1}} \cdot \log\left(n + \sqrt{n^2-1}\right)$

Proved

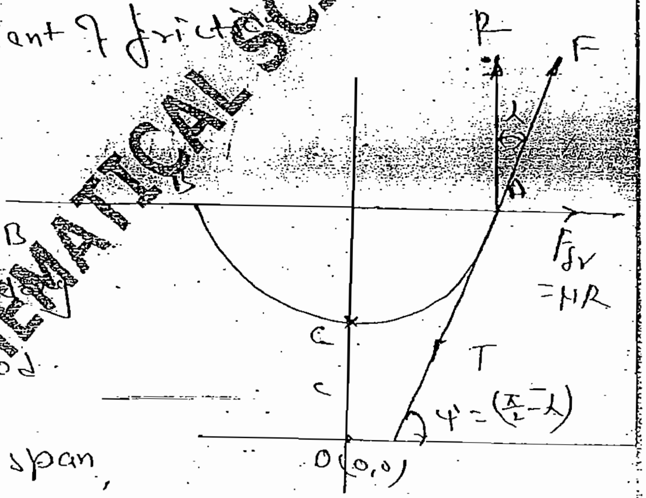
Q.5. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

IAS-2012

$$\mu \log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$$

where, μ is the coefficient of friction.

Sol^n:



Let the end links A and B of a uniform chain slide on a fixed rough horizontal rod.

Let AB is the maximum span, then A and B are in the state of limiting equilibrium.

Let R = Normal reaction of rod at A.

F = resultant force of F_{fr} and R

and $\mu = \tan\lambda$ mean of limiting equilibrium.

For $\rightleftharpoons$, F and T will have the same magnitude but in opposite direction.

Length of the chain, $2s = 2c\tan\psi$

$= 2c\tan\left(\frac{\pi}{2} - \lambda\right)$

$= 2c\cot\lambda$ $\quad (\because \tan\lambda = \mu)$

$\therefore$ length of the chain, $2s = \frac{2c}{\mu}$

Let point $A = (x_1, y_1)$

$\therefore$ maximum span AB

$= 2x = 2 \cdot c \log(\tan\psi + \sec\psi)$

$= 2c \cdot \log\left(\tan\left(\frac{\pi}{2} - \lambda\right) + \sec\left(\frac{\pi}{2} - \lambda\right)\right)$

$= 2c \cdot \log(\cot\lambda + \operatorname{cosec}\lambda)$

$= 2c \cdot \log\left(\cot\lambda + \sqrt{1 + \cot^2\lambda}\right)$

$= 2c \cdot \log\left(\frac{1}{\mu} + \sqrt{1 + \frac{1}{\mu^2}}\right)$

$\therefore$ Required ratio

$$= \frac{\text{Span } AB}{\text{length of chain}} = \frac{2x}{2s}$$

$$= \frac{2c \cdot \log\left(\frac{1}{\mu} + \sqrt{1 + \frac{1}{\mu^2}}\right)}{\frac{2c}{\mu}}$$

$$= \mu \cdot \log\left(\frac{1 + \sqrt{1 + \mu^2}}{\mu}\right)$$

Q.6 The extremities of a heavy string of length $2l$ and weight $2lw$ are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the ring is

$$2l\log(1+\sqrt{2})$$

Hint: $T_x = wc = wl \Rightarrow c = l$ $\therefore x = c\log(\sec\psi + \tan\psi)$

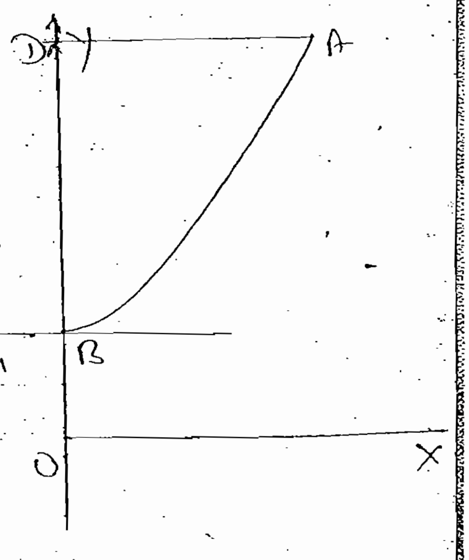
$s = c\tan\psi$ $\quad 2x = 2 \cdot l\log(\sqrt{2}+1)$

$\Rightarrow \tan\psi = \frac{s}{c} = \frac{l}{l} = 1$

$\therefore \psi = 45°$

Q.7 A heavy uniform string, of length l, is suspended from a fixed point A, and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distance between A and B are

$$a\sinh^{-1}(l/a) \quad \text{and} \quad \sqrt{(l^2+a^2)} - a$$



Soln: In this position the arc AB will represent half of the arc of the complete catenary with B as its lowest point.

horizontal force, $F = wa$ $\quad T_0 = F = wa$

i.e. tension at the lowest point

$T_0 = F = wa = wc$

$\Rightarrow a = c$

Let point $A = (x_1, y_1)$

arc $BA = l = s$

From, $s = c \sinh(x/c)$, we have,

$$l = a \cdot \sinh(x_1/a)$$

$$\Rightarrow x = a \cdot \sinh^{-1}(l/a)$$

$\therefore$ Horizontal distance between A and B

$$= a \cdot \sinh^{-1}(l/a) \quad \text{Proved}$$

Again from, $y^2 = s^2 + c^2$ we have,

$$y^2 = l^2 + a^2$$

$$\therefore y_1 = \sqrt{l^2 + a^2}$$

But $y_1 = c + BD$

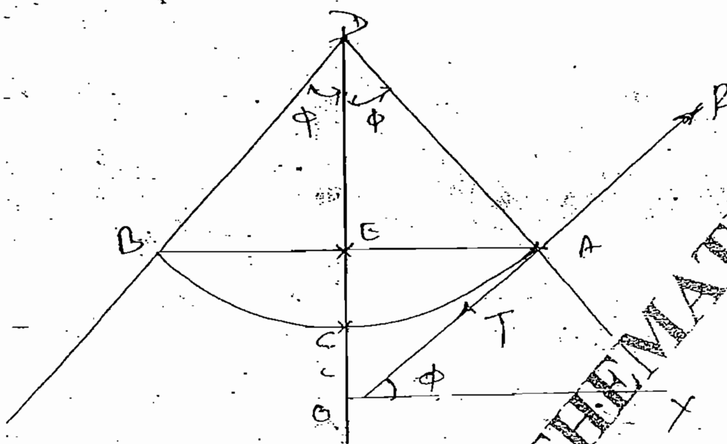
$$\Rightarrow BD = y_1 - c = y_1 - a$$

$$= \sqrt{l^2 + a^2} - a$$

$\therefore$ Vertical distance between A and D is

$$\left(\sqrt{l^2 + a^2} - a\right) \quad \text{Proved}$$

(8) The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles ϕ to the vertical. Prove that the sag in the middle point is $\frac{1}{2}l \cdot \tan\phi/2$.



Hints

for equilibrium of A

$T = R$,

$\therefore \psi = \phi$

at A, $s = \frac{l}{2}$

$\therefore s = c\tan\psi$

$\Rightarrow \frac{l}{2} = c\cdot\tan\phi$

$\Rightarrow c = \frac{l}{2}\cdot\cot\phi$

sag $= EC = OE - OC$

$= y_1 - c$

$= c\sec\phi - c$

$\because y_1 = c\sec\phi$

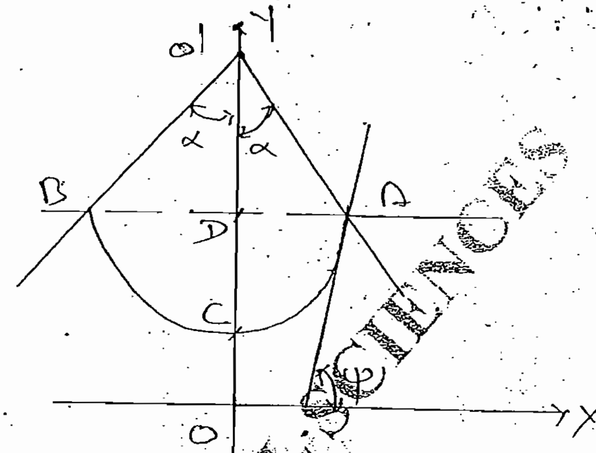
$\therefore y_1 = \frac{l}{2}\cdot\cot\phi\cdot\sec\phi$

(9) A uniform heavy chain is fastened at its extremities to two rings of equal weight, which slide on smooth rods intersecting in a vertical plane, and inclined at the same angle α to the vertical. Find the condition that the tension at the lowest point may be equal to half the weight of the chain and in that case, show that the vertical distance of the rings from the point of intersection of the rod is

$l\cot\alpha \log(\sqrt{2}+1)$

where $2l$ is the length of the chain.

Sol^n:



Let the rods inclined at the same angle 'α' to the vertical intersect at the point O'.

Let A and B be the positions of the rings in ⇌.

Let OC = c

Let w = weight per unit length of the chain.

∴ total weight of the chain = 2wl.

If $T_0 = wc = wl$

$\Rightarrow c = 1$.

Let point A = (x_1, y_1)

From eq^n $s = c\tan\psi$, we have,

$1 = 1.\tan\psi$

$\Rightarrow \psi = \pi/4$

Hence the condition that the tension at the lowest point may be equal to half the weight of the chain is that the tangents at the ends A and B of the chain will make an angle $\pi/4$ to the horizontal.

Now, from $x = c\log(\tan\psi + \sec\psi)$ for the point A, we have,

$$x_1 = 1\cdot\log\left(\tan\tfrac{\pi}{4} + \sec\tfrac{\pi}{4}\right)$$

$$= 1\cdot\log(1+\sqrt{2})$$

i.e. $AD = 1\cdot\log(1+\sqrt{2})$

from $\Delta AO'D$

$O'D = AD\cdot\cot\alpha$

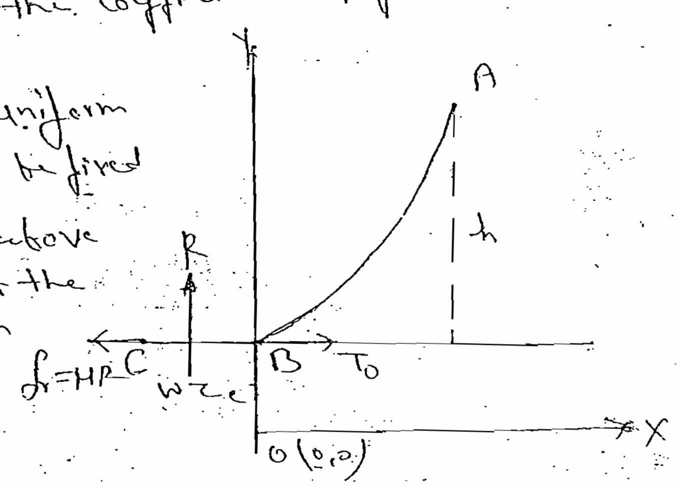
$= 1\cdot\log(1+\sqrt{2})\cdot\cot\alpha$ Proved

(10) A length 'l' of a uniform chain has one end fixed at a height 'h' above a rough table, and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length of the table is

$$l + \mu h - \sqrt{(\mu^2+1)h^2 + 2\mu l h}$$

where μ is the coefficient of friction.

Sol: Let one end of a uniform chain ABC of length 'l' be fixed at A at a height h above the rough table and let the portion BC of the chain rest on the table.

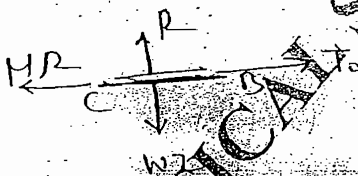


Let $BC = z$

The chain rests in limiting equilibrium with the portion AB in the form of an arc of a catenary with B as its vertex.

weight of the portion $BC = wz$

from figure



$R = wz$

and $\mu R = T_0 = wc$

or, $\mu w z = wc$

or, $c = \mu z$ —(1)

Also, the ~~length of~~ the arc $AB = l - z$ i.e. $s = l - z$

and ordinate of point $A = h + c$

$\Rightarrow y_A = h + \mu z$

From $y^2 = c^2 + s^2$ for the point A, we have,

$(h + \mu z)^2 = (\mu z)^2 + (l - z)^2$

or, $h^2 + \cancel{\mu^2 z^2} + 2h\mu z = \cancel{\mu^2 z^2} + l^2 + z^2 - 2lz$

or, $z^2 - 2(l + \mu h)z + (l^2 - h^2) = 0$

$$\therefore \tau = \frac{2(1+\mu h) \pm \sqrt{4(1+\mu h)^2 - 4(1^2 - h^2)}}{2}$$

$$\text{or, } \tau = (1+\mu h) \pm \sqrt{1^2 + \mu^2 h^2 + 2\mu 1 h - 1^2 + h^2}$$

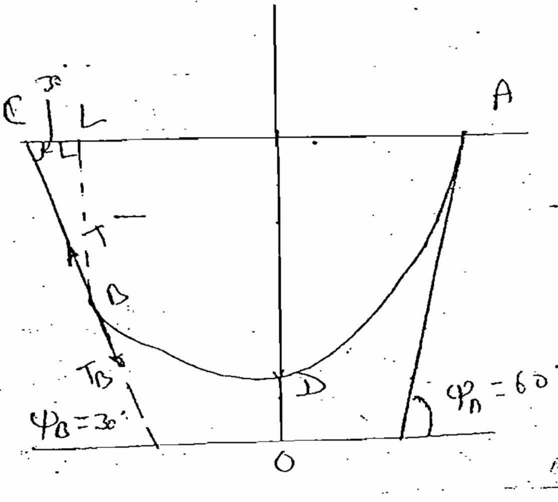
$$= (1+\mu h) \pm \sqrt{(\mu^2+1)h^2 + 2\mu 1 h}$$

for + sign, $\tau > 1$

$$\therefore \tau = (1+\mu h) - \sqrt{(\mu^2+1)h^2 + 2\mu 1 h} \quad \text{Proved}$$

(1) A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The length of the string and chain are such that the ends of the chain at A and B make angles 60° and 30° respectively with the horizontal. Prove that the ratio of their lengths is

$(\sqrt{3}-1):1$



Sol^n Let

l = length of uniform chain AB

a = length of light string BC

the chain AB being heavy will hang in the form of a catenary

while the string BC being light will hang in the form of a straight line.

$\therefore \; T_B = T$

Let D be the lowest point of the catenary.

From eqn $y = c \sec\psi$, we have,

$y_A = c \sec\psi_A = c \sec 60^\circ$

$\Rightarrow y_A = 2c$

and $y_B = c \sec\psi_B = c \sec 30^\circ$

$y_B = \frac{2c}{\sqrt{3}}$

where y_A and y_B are the ordinate of point A and B respectively.

ΔBLC

$BL = BC \sin 30^\circ$

$= \frac{a}{2} \qquad [\because BC = a]$

$\therefore a = 2BL = 2(y_A - y_B) = 2\left(2c - \frac{2c}{\sqrt{3}}\right)$

$\therefore\ a = \frac{4c}{\sqrt{3}}(\sqrt{3}-1)$

If S_1 = length of arc BD

and S_2 = length of arc BA

$\therefore\ S_1 + S_2 = 1$

$\Rightarrow\ 1 = c\tan 60^\circ + c\tan 30^\circ$ $[\because S = c\tan\psi]$

or, $1 = \sqrt{3}\cdot c + c\cdot\frac{1}{\sqrt{3}}$

$= \frac{4}{\sqrt{3}}c$

$\therefore$ Required ratio $= \frac{a}{1} = \dfrac{\frac{4c}{\sqrt{3}}(\sqrt{3}-1)}{\frac{4}{\sqrt{3}}c}$

$= (\sqrt{3}-1):1$ Proved

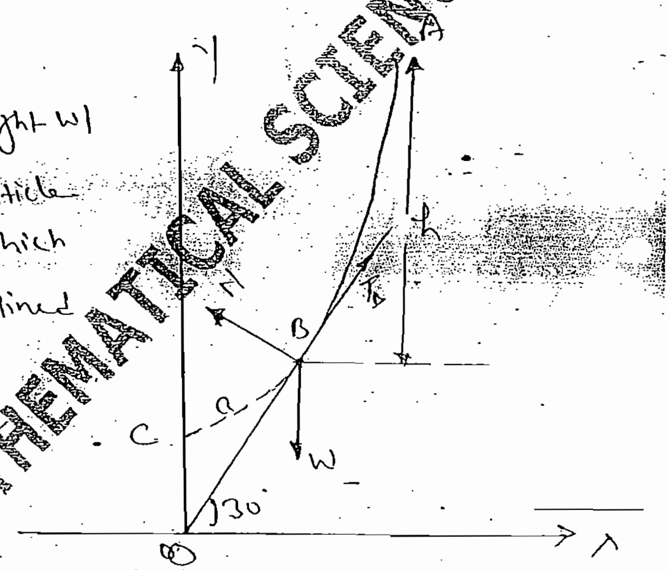
Q. 12 A uniform inextensible string of length l and weight wl, carries at one end B, a particle of weight W which is placed on a smooth plane inclined at 30° to the horizontal. The other end of the string is attached to a point A, ~~situated~~ at a height h above the horizontal through B and in vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the

tangent at B to the catenary laying in the inclined plane if

$$\frac{W}{w} = \frac{(l-h)(l+h)}{(h-\frac{1}{2}l)}$$

Sol^n :

Let AB be the string of weight wl and length l, carrying a particle of weight W at the end B, which is placed on the plane inclined at an angle 30° to the horizontal and the other end is fixed at A, as shown in figure.

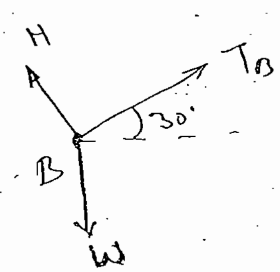


Let C be the lowest point of catenary AB after extending and OX be the directrix.

Let T_B = tension at B, inclined at angle ϕ, 30° to the horizontal.

Coordinate of B = (x_B, y_B)

For particle B
along the inclined plane



$T_B = W.\cos 60°$

$\Rightarrow T_B = \frac{W}{2}$

Now, $y_B = c\sec\psi_B = c\cdot\sec 30^\circ$

$\Rightarrow y_B = \frac{2c}{\sqrt{3}}$

and $T_B = w\cdot y_B = \frac{2wc}{\sqrt{3}}$ — (2)

$\therefore$ from (1) and (2), we have,

$\frac{W}{2} = \frac{2w\cdot c}{\sqrt{3}}$

$\therefore c = \frac{\sqrt{3}}{4}\cdot\frac{W}{w}$ — (3)

Let Arc BC $= a$

Now, from $s = c\tan\psi$, for point B, we have,

$a = c\cdot\tan 30^\circ = \frac{\sqrt{3}}{4}\cdot\frac{W}{w}\cdot\frac{1}{\sqrt{3}}$

$\Rightarrow a = \frac{W}{4w}$ — (4)

For point A

$s_A = l+a, \quad y_A = y_B + h$

from $y^2 = c^2+s^2$, we have,

$(y_B+h)^2 = c^2+(l+a)^2$

and for point B, we have,

$y_B^2 = c^2+a^2$

After subtracting we have,

$(y_B+h)^2 - y_B^2 = (l+a)^2 - a^2$

or, $(2y_B+h)h = (l+2a)l$

or, $h^2 + 2y_B \cdot h = l^2 + 2al$

$\because y_B = \frac{2c}{\sqrt{3}}$ and $c = \frac{\sqrt{3}}{4} \cdot \frac{W}{w}$, $a = \frac{W}{4w}$

$\Rightarrow h^2 + 2 \cdot \frac{2c}{\sqrt{3}} \cdot h = l^2 + 2 \cdot \frac{W}{4w} l$

$\Rightarrow h^2 + \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{4} \cdot \frac{W}{w} \cdot h = l^2 + \frac{W}{2w} l$

$\Rightarrow \frac{W}{w}\left(h - \frac{l}{2}\right) = l^2 - h^2$

$\Rightarrow \frac{W}{w} = \frac{l^2 - h^2}{h - l/2}$

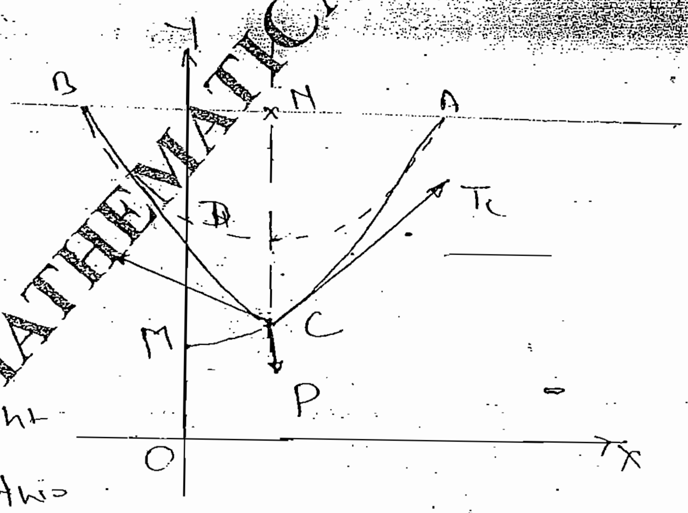
$= \frac{(l+h)(l-h)}{(h - l/2)}$

Proved

Q.13. IFoS-2010

A uniform chain of length $2l$ and weight W is suspended from two points A and B, in the same horizontal line. A load P is now suspended from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is $\frac{1}{2}\left\{P\frac{l}{h} + W\cdot\frac{h^2+l^2}{2hl}\right\}$

Sol:



Let a uniform chain of length '$2l$' and weight W is suspended from two points A and B in the same horizontal level freely under gravity in the form of catenary ADB.

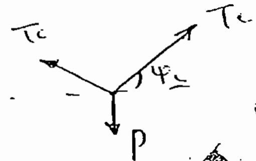
When a load P is attached at D of the chain, this will come down to 'C' as shown in figure and then the two portions AC and BC of the chain each of length l will be the part of two equal catenaries.

Let M be the lowest point and OX be the directrix of the catenary of which AC is an arc.

the weight per unit length of the chain

$$w = \frac{W}{2l}$$

<u>For point C</u> :



$$2T_c \cdot \sin\psi_c = P \quad -(1)$$

But $T_c \sin\psi_c = wa$

where $a = \text{arc } MC$

from (1), $wa = P/2$

$$\Rightarrow a = \frac{P}{2w} = \frac{2Pl}{2W} \quad -(2)$$

Let point $A = (x_A, y_A)$

and point $C = (x_c, y_c)$

at A , $S = S_A = \text{arc } MA = \text{arc } MC + \text{arc } CA$

$= a + l$

and $y = y_A$

at ~~point C~~

$s = s_c = \text{arc } MC = a$

and $y = y_c$

and Sag, $CM = h$ (given)

$\therefore \; y_c + h = y_A$

$\Rightarrow y_c = y_A - h$

from $y^2 = c^2 + s^2$, we have

$y_A^2 = c^2 + (a+l)^2$

and $y_c^2 = c^2 + a^2$

After subtracting, we have,

$y_A^2 - y_c^2 = (a+l)^2 - a^2$

$= (2a+l)\,l$

or, $y_A^2 - (y_A - h)^2 = (2a+l)\,l$

or, $(2y_A - h)\,h = 2al + l^2$

or, $y_A = \frac{1}{2}\left\{ h + \frac{2al + l^2}{h} \right\}$

Hence, each of terminal tension is given by

$T = T_A = T_B = w y_A = \frac{W}{2l} \cdot \frac{1}{2}\left\{ h + \frac{2al+l^2}{h} \right\}$

$$\therefore T = \frac{W}{4l}\left\{ h + \frac{2l \cdot \frac{2Pl}{2W} + l^2}{h} \right\}$$

$$= \frac{W}{4l}\left\{ h + 2 \cdot P \cdot \frac{l^2}{Wh} + \frac{l^2}{h} \right\}$$

$$= \frac{1}{2}\left\{ P \cdot \frac{l}{h} + W \frac{h^2 + l^2}{2hl} \right\}$$

Proved

Q.14 A uniform chain of length 2l and weight 2W, is suspended from two points in the same horizontal line. A load W is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h. Show that the terminal tension is $\frac{1}{2} W \cdot \frac{h^2 + 2l^2}{hl}$

Q.15. A heavy string of uniform density and thickness is suspended from two given points in the same horizontal plane. A weight an n^{th} that of the string is attached to its lowest point. Show that if θ, ϕ be the inclination to the vertical of the tangent at the highest and lowest points of string $\tan\phi = (1+n)\tan\theta$

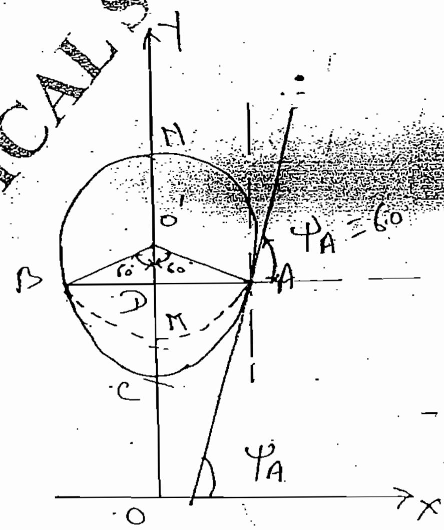
IFoS-2006

Q.16 Show that the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with two-thirds of the circumference of the pulley is

$$a\left\{\frac{3}{\log(2+\sqrt{3})}+\frac{4\pi}{3}\right\}$$

Sol^n:

Let AMBMA be the circular pulley of radius a and AMBCA the endless chain hanging over it.



Total length of the chain

$= \frac{2}{3}$(circumference of the pulley) + arc BCA

$= \frac{2}{3}(2\pi a)$ + arc BCA

$= \frac{4}{3}\pi a$ + arc BCA —— (1)

Here, ACB forms an catenary, with C as a vertex.

Let $OC = c$

from $\Delta\, G'AD$, we have,

$$DA = GA \sin 60^\circ = a \cdot \frac{\sqrt{3}}{2}$$

And from $x = c \log(\sec\psi + \tan\psi)$

for the catenary ACB

at point A, $\psi_A = 60^\circ$

$$\therefore x = c \log(\sec 60^\circ + \tan 60^\circ)$$

$$= c \log(2 + \sqrt{3})$$

But $x = DA = a\frac{\sqrt{3}}{2}$

$$\therefore c = \frac{a\sqrt{3}}{2 \log(2+\sqrt{3})}$$

Now from $S = c \tan\psi$

at point A,

$$S_A = c \tan 60^\circ$$

$$= \frac{a \cdot \sqrt{3}}{2 \log(2+\sqrt{3})} \sqrt{3}$$

$\therefore$ total length of arc ACB $= 2S_A$

$$= \frac{3a}{\log(2+\sqrt{3})}$$

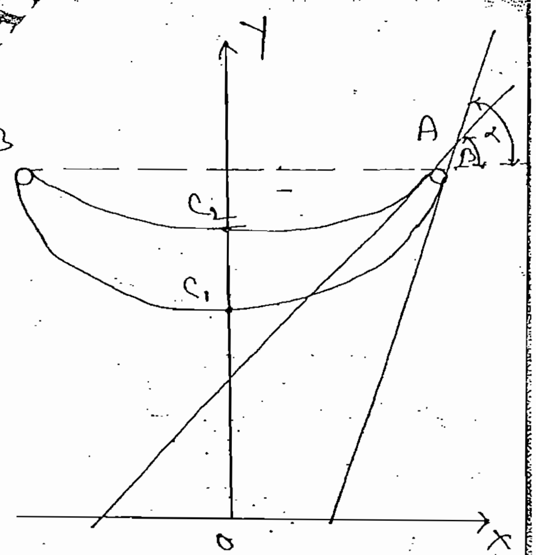
$\therefore$ from (1)

Total length of the chain

$$l = \frac{4}{3}\pi a + \frac{3a}{2\log(2+\sqrt{3})}$$

$$= a\left\{\frac{4}{3}\pi + \frac{3}{\log(2+\sqrt{3})}\right\}$$

Q.17 An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of ~~the distance~~ between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs.



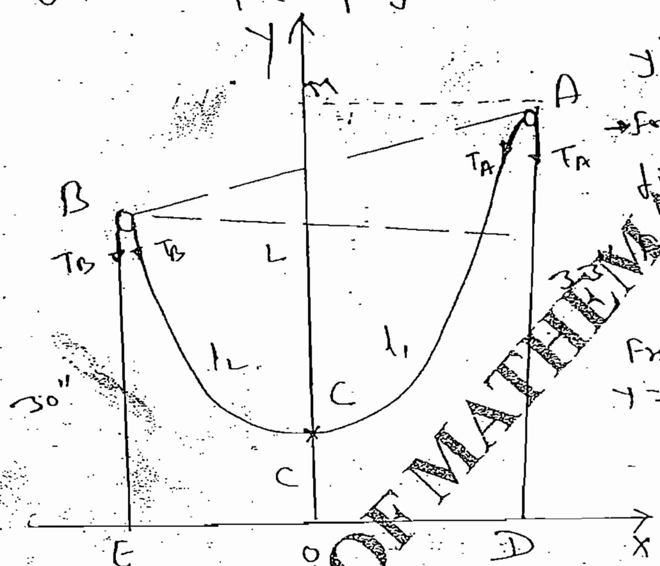
i.e. $\frac{C_2 - C_1}{S_1 + S_2} = \tan\left(\frac{\alpha-\beta}{2}\right)$

Hint: $T_{A_1} = T_{A_2}$, $S_1 = c_1\tan\psi_1$, $S_2 = c_2\tan\psi_2$ $\quad\therefore y_{1A} = y_{2A}$

$wy_{1A} = wy_{A_2}$ $\quad c_1\tan\psi_1 = c_2\tan\psi_2$

$\therefore y_{1A} = y_{2A}$

(18) A heavy uniform string, 90" long, hangs over two smooth pegs at different heights. The parts which hangs vertically are of length 30" and 33". Prove that the vertex of catenary divides the whole string in the ratio of 4:5 and find the distance between the pegs.



$y^2 = c^2 + s^2$ → for point A & B find l_1 & l_2

For point A

$T_A|_{AC} = T_A|_{AD}$

$w \cdot y_A = w \cdot AD$

$\Rightarrow y_A = AD$ ← height of point A above the directrix

Similarly for point B

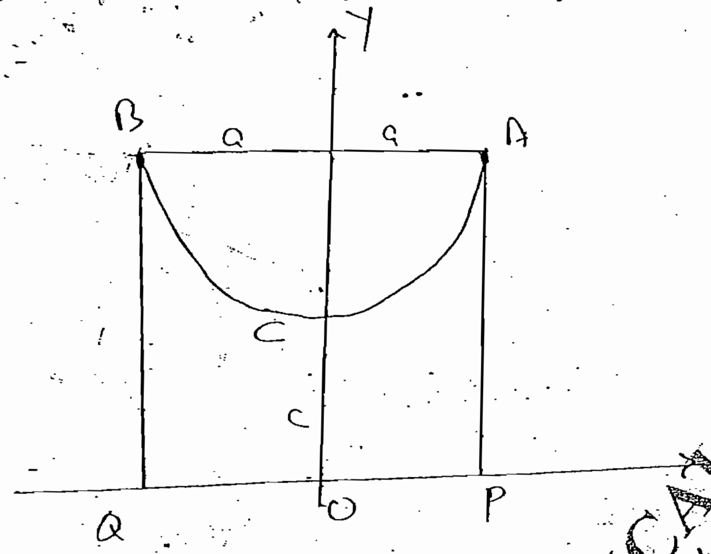
$l_1 + l_2 = 27$

From $y_A^2 = c^2 + l_1^2$ find c

From $y = c \cosh \frac{x}{c}$

Q.(19) A string hangs over two smooth pegs which are at the same level. Its free ends hanging vertically. Prove that when the string is of shortest possible length the parameter of the catenary is equal to half the distance between the pegs, and find the whole length of the string.

Sol:



Suppose A and B are two smooth pegs at the same level and at a distance 2a apart

i.e. $AB = 2a$

A string hangs over the pegs A and B. The portion AP and BQ of the string hang vertically and the portion ACB is in the form of catenary with vertex C and directrix is OX.

The pegs are smooth

$\therefore$ tension at point A, due to string AP = tension at the point A due to string ACB

$\Rightarrow w \cdot AP = w \cdot y_A$ [where $w = \frac{\text{weight}}{\text{length}}$]

$\Rightarrow AP = y_A$

$\therefore y_A = AP$

Show that the point P lies on the directrix OX of the catenary ACB.

Similary, the other free end Q of the string also lies on the directrix OX.

Let. c = parameter of Catenary.

at point A,

$$y = y_A = AP \text{ and } x = x_A = a$$

$$\therefore \quad s = \text{arc } AC = c \sinh\left(\frac{a}{c}\right)$$

and $$y_A = AP = c\cosh(a/c)$$

$\therefore$ total length of the string

$$l = 2\times(\text{arc } AC + AP)$$

$$\Rightarrow l = 2\times\left(c\cdot\sinh(a/c) + c\cdot\cosh(a/c)\right)$$

$$= 2c\left(\frac{e^{a/c} - e^{-a/c}}{2} + \frac{e^{a/c} + e^{-a/c}}{2}\right)$$

$$= 2\cdot c\cdot e^{a/c} \quad \text{——} \quad (1)$$

Here, $l = f(c)$ only

∴ For maximum or minimum value of 'l', we have,

$$\frac{dl}{dc} = 0$$

$$\Rightarrow 2\cdot e^{a/c} + 2\cdot c\cdot e^{a/c}\left(\frac{a}{-c^2}\right) = 0$$

$$\Rightarrow 2\cdot e^{a/c}\left(1 - \frac{a}{c}\right) = 0$$

$$\because e^{a/c} \neq 0$$

$$\therefore a = c$$

Now, $$\frac{d^2l}{dc^2} = -\frac{2a}{c^2}\cdot e^{a/c}\left(1 - a/c\right) + 2\cdot e^{a/c}\cdot\frac{a}{c^2}$$

$$= 2\cdot\frac{a^2}{c^3}\cdot e^{a/c}$$

$$\therefore \frac{d^2l}{dc^2} > 0 \quad \text{for } c = a.$$

∴ l is minimum, when $c = a$

Thus when the string is of shortest possible length, we have

$$c = a = \frac{1}{2}(2a)$$

$$= \frac{1}{2} \times \text{distance between the pegs}$$

Proved

$\therefore$ whole length of String, from (i),

$l = 2ca$ [on putting $c = a$]

Ans